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MATHEMATICS MAGAZINE

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IN MEMORIUM

It is with a feeling of deep personal sorrow and great professional loss that we announce the death of Prof. A.D. Michal. He has been one of the cornerstones of the Mathematics Magazine since its inception. A memorial biographical sketch of Prof. Michal will appear in a future issue.

A DEVELOPMENT OF ASSOCIATIVE ALGEBRA AND AN ALGEBRAIC THEORY OF NUMBERS, II

H. S. Vandiver

INTRODUCTION

In a previous article under the present title which will be referred to here as (I), we considered an infinite ordered set of symbols

$$(1) \qquad C_1,\ C_2,\ C_3,\ \ldots$$

in which if we denote a particular element by C_k, its immediate successor in this is $C_{k'}$, where k denotes a natural number and k' its immediate successor in the set of natural numbers. We then introduced in addition to these symbols the symbol + (called a plus sign); x (called a multiplication sign); and (, called a left parenthesis symbol; and), called a right parenthesis symbol. We next set up a system of postulates covering *combinations* of the symbols just mentioned and the idea of equality (=) between these symbols. We assume nothing concerning equality among the symbols (1) themselves and using the and using the postulate and various assumptions conserning equality and the elements in (1), we obtained elementary arithmetic of the of the natural numbers and certain finite arithmetics. In particular we derived finite arithmetics in which the cancellation law of addition did not hold. Another feature of the development in (I) is the fact that we do not use the symbol of equality to mean "is" or the symbol $\neq$ to mean "is not," so that we were obliged to introduce the postulate of substitution to take care of a possible generalization of ordinary equality. Also, let A, B, C, and D denote combinations of the type indicated above. Let us then symbolize the statement "If $A = B$, then $C = D$" by

$$A = B \rightarrow C = D.$$

This whole finite linearly ordered set of symbols may be regarded as a combination. In particular, an equation such as, for example,

$$(a + b) + c = m + n$$

may be regarded as a combination, and we might replace ordinary language about $4 + 2 = 6$ and $4 + 2 = 7$ by stating that the first combination is a preferred one and the second is non-preferred.

In the present paper we develop still further the possible generalizations of the usual notion of equality.

Cauchy[1], in attempting to justify the use of complex numbers in ordinary algebra, employed polynomials with real coefficients in an indeterminate x, and noted that, if $i^2 = -1$, then, cf. also the statements following Theorem 5 of the present paper,

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

corresponds to

$$(a + bx) + (c + dx) \equiv (a + c) + (b + d)x \qquad (\text{mod } x^2 + 1).$$

Now there is certainly a type of correspondence here, as Cauchy indicates, since $a + bx$ corresponds to $a + bi$, etc., and *equality corresponds to congruence modulo* $(x^2 + 1)$ Also, the distinct complex numbers correspond, one to one, to the incongruent residues modulo $(x^2 + 1)$. Dedekind[2] made use of the idea of classes of residues modulo m so that each integer a defines a unique class of integers ("Zahlklasse") of the form $a + km$, k being an integer. E. H. Moore[3] similarly used the idea of residue classes with respect to a modular system defined by a prime p and $f(x)$, where the latter is a polynomial in x with integral coefficients, and showed that these classes form a field. Steinitz[4] used residue classes with respect to the modulus $g(x)$, where $g(x)$ has rational coefficients and is irreducible in the rational field. He employed this to set up his theory of algebraic fields. (Here $f(x)$ is also irreducible, modulo p).

The above notions with the exception of Cauchy's always seemed to the writer to be unnecessarily involved. It seems a bit less complicated to speak of the set of incongruent integers modulo m, and of the group formed by them under addition modulo m, instead of setting up the notion of class and defining the equality of classes and the addition of classes.

Without confining himself to the idea of residue classes, Kronecker indicated how to adjoin the negative rational integers to the set consisting of zero and the positive rational integers by using a modulus $x + 1$. Otherwise he followed the point of view of Cauchy exactly. He also indicated how to introduce the rational numbers by employing moduli of the form $xa - 1$, where a denotes a rational integer. He was followed by Wedderburn[5], who used the idea of Cauchy also along

[1] *Oeuvres*, 1st series, 10, 317-319.

[2] *Werke*, Bd. III, 74-76. He gives the idea for the more general case of algebraic numbers.

[3] *Trans. New York Math. Soc.*, May 1893; cf. also Dickson, "Linear Groups," *Teubner*, 1901, 1-7; and Weber, *Algebra*, 2nd. ed., Bd. II, 60-61; 305-306.

[4] *Journal fur Mathematik*, vol. 137, 193-194, 1910.

[5] "Algebraic Fields," *Annals of Math.* (2), vol. 24, 237-264, 1923.

these lines except that, instead of starting with positive rational integers, he employed a certain type of semi-field in which subtraction and division were not defined and showed how to adjoin the necessary elements in order to carry out these operations.

The point of view we want to use in this paper in connection with adjunction makes it necessary for us to define the idea of *indeterminate* in a particular way. It must be defined initially such that no numbers or quantities are involved except the natural numbers. Thus, we cannot use the familiar notion of a transcendental adjunction. This would involve the conception of a quantity not in the set of natural numbers. In this connection Wedderburn (l.c.) started off with a system which he called a semi-field F, and defined two polynomials in an indeterminate x with coefficients in F as being equal if, and only if, they were equal when x was replaced by any element of F.

However, this definition makes it necessary to treat as exceptional the case where F is a finite field since $x^{p^n} = x$ for each element in the finite field of order p^n. Our definition would appear to be an improvement over Wedderburn's, as we do not need any additional discussion of elements in finite fields.

In the first part of this introduction we have referred to a postulate of substitution. However, in (I) we set up an algebra in which this substitution law did not hold but other postulates, such as the associative laws, held. Here we shall elaborate on this idea. Suppose we fix consecutive unit elements of distance along a straight line as indicated below

0 1 2 3 4 5 6 . . . .

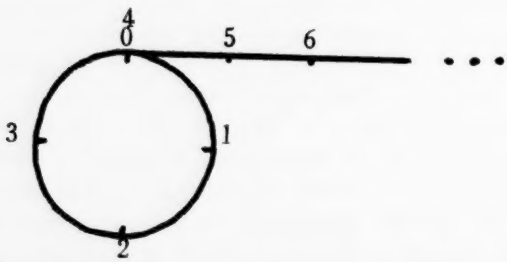
The symbol P_a indicates the operation of passing over a units of distance. We shall define addition among the P symbols and write

$$P_a + P_b = P_{a+b}$$

and in general the result of carrying out this operation or more complicated ones involving addition is equal to any other combination of the P's involving addition provided that if we carry out the operations, we reach the same point on our line; that is, for example,

$$P_4 + P_2 = P_5 + P_1.$$

It is clear from this that the following laws, true in elementary arithmetic, hold also for the P's: the commutative law, the associative law, the substitution law, and the closure law. But suppose now we consider the figure that we get by replacing a line by the figure as follows, using the same units of measurements:



We use the symbol P_a' to indicate the operation of passing along the outline of this figure a units, and specify that $P_h' = P_k'$ if and only if the result of each of the two operations indicated leads us to the same point on our figure. Obviously all the laws just mentioned concerning the P's will carry over to the P''s *except the law of substitution since*

$$P_0' + P_1' = P_0' + P_1'$$

and

$$P_4' = P_0';$$

yet, we cannot substitute P_4' for P_0' in the left-hand member of the first relation, as this gives us P_5' instead of P_1'.

Since we have used only addition in the above, this indicates a system of single composition where the law of substitution does not hold. However, if we define a second operation, multiplication, as follows

$$P_a' \times P_b' = P_{ab}'$$

then we have a system of double composition where the substitution law also breaks down.

Throughout the rest of this paper, we often state well known theorems without proofs, particularly when proofs may be found in a number of books on abstract algebra. What seem to be the lesser results are often stated as problems. We devote most of our attention, as far as demonstrations are concerned, either to explaining new proofs of old results or to proving new results. New concepts, however, will be discussed at length.[6]

[6] The material which I shall consider in the balance of the present paper was developed mainly during seminars on abstract algebra and number theory which I gave at the University of Texas during the last 20 years. During this period I also discussed some of the ideas in personal conversations with various mathematicians. As a result of this, I am indebted for suggestions and corrections to R. D. Allentharp, F. C. Biesele, A. Church, J. L. Dorroh, O. B. Faircloth, H. C. Miller, J. B. Rosser, J. M. Slye, and W. J. Viavant. In particular, several preliminary drafts of the present paper were discussed in a seminar at the University of Texas which was attended by Anne Breese Barnes, William T. Guy, Richard P. Kelisky, William C. Long, Charles A. Nicol, Ernest T. Parker, and Milo W. Weaver, and their criticisms were most helpful. Also, the last mentioned detected serious errors in the first draft of the article.

On p. 249 of (I) it was stated that semi-groups and semi-rings would be discussed in the present article, but considerations of length necessitated a change in plans, and these topics will be discussed in a later paper.

1

ADJUNCTION OF ZERO, NEGATIVE INTEGERS AND RATIONAL FRACTIONS TO THE SET OF NATURAL NUMBERS

We shall first show how to adjoin zero to the set of natural numbers. To do this we use the properties of the natural numbers described in (I), some of which we shall state separately here, as they will be extensively employed.

Postulate 1: If e and f denote natural numbers, neither = 1, then the statements

$$e = f, \quad e > f, \quad \text{and} \quad e < f$$

are mutually exclusive, that is, one and just one of these relations holds. Also, one and just one of the relations $g = 1$, $g > 1$ holds. If a, b and c denote natural numbers and if $a + b = c$ we write $a < c$ and/or $c > a$. If we introduce one of the three statements

$$\text{(A)} \qquad a > b, \quad a < b, \quad \text{and} \quad a = b$$

and employ it in connection with our postulates and established theorems, and derive any two of the relations

$$\text{(B)} \qquad d = e, \quad d > e, \quad d < e,$$

where d and e denote natural numbers, this is said to be a contradiction; and the particular one of the three relations (A) which we introduced is said to be false.

We assume the postulate of mathematical induction. Also, if all the small letters denote natural numbers, then we have

Theorem 1. *If:*

$$a + c = b + c,$$

then

$$a = b.$$

Theorem 2. *If:*

$$ac = bc,$$

then

$$a = b.$$

We may also prove the fundamental (not proved in (I))

Theorem 3. *If A denotes a combination, then we may determine a natural number denoted by a such that*

$$A = C_a,$$

where the righthand member is defined in (1) *of the present paper.*

The notion of congruence in the set of natural numbers will now be introduced.

Definition: If

(2) $$ms_1 + r_1 = ms_2 + r_2,$$

we say that r_1 is congruent to r_2 modulo m, and this statement will be symbolized by

(3) $$r_1 \equiv r_2 \pmod{m}.$$

Also, (3) implies the existence of natural numbers s_1 and s_2 such that (2) holds.

LEMMA I. If $r_1 \equiv r_2 \pmod{m}$, $r_1 < m$, $r_2 < m$, then $r_1 = r_2$; also, the statement $ms_3 + r_3 = mq_2$, $r_3 < m$ leads to a contradiction.

To prove the second part of the lemma, let us assume that

(4) $$ms_1 = ms_2 + t; \quad t < m.$$

If $s_1 > s_2$, we have by definition $s_1 = s_2 + k$. We then obtain, by substitution,

$$m(s_2 + k) = ms_2 + t.$$

This gives

$$ms_2 + mk = ms_2 + t,$$

or, by Theorem 1,

$$mk = t.$$

Now, if $k = 1$, the above relation is a contradiction, for $t = m$, whereas $t < m$ by assumption. If $k > 1$, then by definition $k = 1 + h$. Employing substitution, we have

$$m(1 + h) = t,$$

or

$$m + mh = t.$$

Here $t > m$ which is a contradiction, for $t < m$ by assumption. If $s_1 = s_2$ or $s_1 < s_2$ we are led, by methods similar to those used above, to contradictions.

To prove the first part of the lemma, assume that

$$ms_1 + r_1 = ms_2 + r_2.$$

If $r_2 > r_1$, then by definition

$$r_2 = r_1 + t.$$

Using substitution, we obtain

$$ms_1 + r_1 = ms_2 + r_1 + t$$

which gives, using Theorem 1, the relation (4), which has already been shown to be impossible. Similarly, the assumption $r_2 < r_1$ leads to a contradiction. Hence $r_2 = r_1$ and the lemma is proved.

Denoting the set of natural numbers by N, we now introduce the concept of polynomials in N. Let x denote any natural number. We first discuss the idea of exponents and powers and denote $x\ x \cdots x$ by x^n, where there are n terms in the first ordered set, n being called an exponent of x in the expression x^n, and x^n is said to be of degree n. The two relations, where all the symbols denote elements in N,

$$x^a x^b = x^{a+b}, \quad (x^a)^b = x^{ab},$$

are given without proof.

In (I) we built up the arithmetic of the natural numbers starting with the C's in (1) of the present paper. In this way we obtained for the natural numbers (not all the proofs are given in full) the commutative laws of addition and multiplication, the associative laws of addition and multiplication, the distributive law, and $a \times 1 = a$. Now, a peculiarity of all these laws is the fact that *they hold for any natural numbers,* for example, $a + b = b + a$, if a, b each denote any natural numbers. On the other hand, in the general closure laws of addition and multiplication for the natural numbers this is not the case. For example, in the statement, "If a and b denote natural numbers, then there is a natural number denoted by k such that $a + b = k$", evidently k depends on a and b.

We shall now use a natural number which we shall call x and confine ourselves to obtaining relations involving x which are to hold for whatever natural numbers we substitute for x. Hence, proceeding in this way, we can obtain the following relations:

$$x^3 + 2x^3 = (2+1)x^3 = 3x^3; \quad 3x^2 \cdot 5x^3 = 15x^5.$$

We note that each of these hold for any natural number in place of x. This, of course, is because the powers of x involved do not appear in any of the applications of the closure laws of addition and multiplication we have used. In a similar way this follows:

$$(x+2)(2x^2+3) = 2x^3 + 4x^2 + 3x + 6.$$

We are led, in this way, to consider expressions like

Type I. $\quad b_1 x^{d_1} + b_2 x^{d_2} + \cdots + b_n x^{d_n}, \quad d_1 < d_2 \cdots < d_n;$

Type II. $\quad b' + b_1' x^{e_1} + b_2' x^{e_2} + \cdots + b_k' x^{e_k}, \quad e_1 < e_2 \cdots < e_k;$

Type III. $\quad b,$

where the b's, b''s, d's, and e's denote elements of N. Such an expression will be called a polynomial in N, and x will be called an indeterminate in N. Also d_n is the degree of the Type I polynomial; e_k is the degree of the Type II polynomial. For the present, we shall not define the degree of Type III. The b's and b''s in Type I and Type II are called coefficients.

Now we shall define congruences involving these polynomials. Let

$$A(x) + f(x)G(x) = B(x) + f(x)H(x),$$

where $A(x)$, $f(x)$, $G(x)$, $B(x)$ and $H(x)$ are each polynomials, then $A(x)$ is said to be congruent to $B(x)$ with respect to the modulus $f(x)$, and this last statement is said to be equivalent to

$$A(x) \equiv B(x) \pmod{f(x)}.$$

Conversely, we state that if the last relation holds, then the polynomials $G(x)$ and $H(x)$ exist so that the relation just preceding the last also holds. We now consider polynomials which are congruent with respect to the modulus x, or as we shall say modulo x. We note that

$$ax + x \times 1 = x + ax$$

and

$$a + x + x \times 1 = a + 2x;$$

hence, using the definition of congruence, we may write

$$ax \equiv x \pmod{x}; \quad x^2 \equiv x \pmod{x}$$

and

$$a + x \equiv a \pmod{x}.$$

We note that there is a correspondence between these relations and the relations of ordinary arithmetic to the effect that

$$a \times 0 = 0, \quad 0 \times 0 = 0 \quad \text{and} \quad a + 0 = a,$$

and we have equality corresponding to congruence modulo x. *We shall now show that if we adjoin zero to the set of natural numbers and give it the properties just mentioned, that there is what we shall call an isomorphism*[7] *between the two systems.* To carry this out we need to show that, if a_1 and a_2 denote natural numbers,

$$a_1 \equiv a_2 \pmod{x}$$

gives

(5) $$a_1 = a_2,$$

also if a denotes a natural number, that

$$a \equiv x \pmod{x}$$

leads to a contradiction. By the definition of congruence we may write

[7] This notion is introduced informally at this stage. The complete definition involves the requirements stated in our Problem 1. We plan to give in our next article the formal definition for the system we shall call a semi-ring (which includes N).

$$a_1 + xh(x) = a_2 + xk(x). \tag{6}$$

We now introduce an additional idea in connection with these polynomials, that is, we shall fix on a particular natural number for x and note that if we employ the closure laws of addition and multiplication for a polynomial involving this particular x that the polynomial equals a natural number for any of the three types of the polynomials mentioned, as we see from Theorem 3. In view of this, we shall assume a natural number x such that $x > a_1$ and $x > a_2$ and (6) gives

$$a_1 + bm_1 = a_2 + bm_2,$$

where we replace this particular x by b. If we assume $a_1 > a_2$ and employ the cancellation law of addition, we obtain a contradiction in view of our lemma. Hence, $a_1 = a_2$ in (5).

Now assume

$$x \equiv a \pmod{x}.$$

This gives

$$x + xr(x) = a + xt(x).$$

In this relation, set $x = a + 1$. We obtain

$$1 + (a+1)b_1 = (a+1)b_2$$

which is again a contradiction in view of our lemma.

From the above it then follows that there is an isomorphism between the set consisting of x and N with the set zero and N, where in the first instance we are using congruence modulo x and in the second instance equality as the symbols of relation. In the correspondence, the natural number a in one of the sets corresponds to the natural number a in the other; also, zero in the second system corresponds to x in the first. Under addition and multiplication we have the correspondences:

$$\begin{cases} a + 0 = a \\ a + x \equiv a \pmod{x}, \end{cases} \quad \begin{cases} 0 + 0 = 0 \\ x + x \equiv x \pmod{x}, \end{cases} \quad \begin{cases} a + b = c \\ a + b \equiv c \pmod{x}. \end{cases}$$

$$\begin{cases} a \times 0 = 0 \\ a \times x \equiv x \pmod{x}, \end{cases} \quad \begin{cases} 0 \times 0 = 0 \\ x \times x \equiv x \pmod{x}, \end{cases} \quad \begin{cases} a \times b = c \\ a \times b \equiv c \pmod{x}. \end{cases}$$

In view of these correspondences we say that zero may be adjoined to the set of natural numbers. Denote this enlarged set by $N[0]$. Also, in general, if k denotes an element belonging to a set S, we shall from now on symbolize this by $k \in S$. If $a \in N[0]$, $b \in N$, and $a + b = n$, then we write $a < n$ or $n > a$. Conversely, if $a < n$ or $n > a$, then there exists an element denoted by b, $b \in N$ such that $a + b = n$ with $b \in N$.

Problem 1. Prove that the postulates 1, 2, 3, 5 and 6 of our paper (I), p. 243, hold also if we replace the combinations referred to therein by expressions in which the C's appearing in the combinations are replaced by symbols denoting elements in $N[0]$.

Problem **1a.** Prove that one and only one of the following relations holds; if a and b belong to $N[0]$:

$$a > b, \quad a < b, \quad a = b.$$

Show also that $0 < c$, for $c \in N$.

We shall now show how to adjoin the negative integers to $N[0]$: We first introduce the idea of indeterminate in connection with our enlarged system $N[0]$. It is defined as it was when elements of N were involved except that we may also replace x, in any of our relations involving polynomials, by zero and employ, if desired, $x \cdot 0 = 0$ since this holds for any $x \in N[0]$. We next note that we may extend the definition of congruence that we gave, in connection with N, to $N[0]$, so that if $m \in N$, and

$$ms_1 + r_1 = ms_2 + r_2$$

we may write

$$r_1 \equiv r_2 \pmod{m}$$

with s_1, s_2, r_1 and r_2 each belonging to $N[0]$ and from the last relation we may deduce an equation of the first type. Then lemma I gives the result that if $r_1 \equiv r_2 \pmod{m}$ with $r_1 < m$ and $r_2 < m$, $r_1 \in N[0]$, $r_2 \in N[0]$, $m \in N$, then $r_1 = r_2$.

From

$$x^2 + (x + 1) \times 1 = 1 + x(x + 1)$$

with x an indeterminate in $N[0]$, there is obtained

(7) $$x^2 \equiv 1 \pmod{x+1}.$$

Assume

(8) $$a + bx \equiv a_1 + b_1x \pmod{x+1}.$$

It will be seen that this gives

(9) $$a + b_1 \equiv a_1 + b \pmod{x+1},$$

when we add $b_1 + b$ to both members. We may then write

$$a + b_1 + (x + 1)A(x) = a_1 + b + (x + 1)B(x),$$

where $A(x)$ and $B(x)$ denote polynomials in x with coefficients $\in N[0]$. We now select a particular x so that $x + 1 > a + b_1$ and $x + 1 > a_1 + b$. Then, if $x + 1 = m$, we obtain

$$a + b_1 \equiv a_1 + b \pmod{m};$$

and by the lemma we obtain

(10) $$a + b_1 = a_1 + b.$$

We wish to show a correspondence between

$$(C) \qquad \left.\begin{array}{rrrrr} & x, & 2x, & 3x, & \cdots \\ 0, & 1, & 2, & 3, & \cdots \end{array}\right\}$$

and the set

$$(D) \qquad \left.\begin{array}{rrrrr} & -1, & -2, & -3, & \cdots \\ 0, & 1, & 2, & 3, & \cdots \end{array}\right\}$$

where $ax \longleftrightarrow -a$, a in N, $b \longleftrightarrow b$, b in $N[0]$. It will first be proved that the elements of the first set are incongruent modulus $x+1$ which will correspond to the idea that the elements of the second set are unequal or distinct. Since (8) yields (10), we set $a = 0$ and $a_1 = 0$ in (8), and obtain from (10) $b_1 = b$. Also, if we assume, with b_1 in N and a in $N[0]$,

$$a \equiv b_1 x \pmod{x+1}$$

in (8), we obtain by setting $b = 0$ and $a_1 = 0$ in (10)

$$a + b_1 = 0,$$

with b_1 a natural number, which is a contradiction by the statement in Problem 1. Further, if $a \equiv a_1 \pmod{x+1}$ with $a \in N[0]$, $a_1 \in N[0]$ then (10) gives, if we set $b = 0$ and $b_1 = 0$, the result $a = a_1$.

Under addition and multiplication these correspondences follow through as follows, if we omit the statement of the identical correspondences of $N[0]$ with $N[0]$:

If $a, c, d, k \in N[0]$; $b, e \in N$, $(-0) = 0$

$$\left\{\begin{array}{l} a + (-b) = (-d), \\ \left\{\begin{array}{l} a + bx \equiv dx \pmod{x+1}, \\ \text{if } \ b = a + d. \end{array}\right. \end{array}\right. \qquad \left\{\begin{array}{l} a + (-b) = c, \\ \left\{\begin{array}{l} a + bx \equiv c \pmod{x+1}, \\ \text{if } \ a = b + c. \end{array}\right. \end{array}\right.$$

$$\left\{\begin{array}{l} (-a) + (-c) = (-k) \\ \left\{\begin{array}{l} ax + cx \equiv kx \pmod{x+1}, \\ \text{if } \ a + c = k. \end{array}\right. \end{array}\right.$$

$$\left\{\begin{array}{l} (-e)(-b) = eb \\ ex \cdot bx \equiv eb \pmod{x+1} \end{array}\right. \qquad \left\{\begin{array}{l} a(-b) = -ab \\ a \cdot bx \equiv x \cdot ab \pmod{x+1} \end{array}\right.$$

Consequently, we have an isomorphism between the sets (C) and (D); and we can adjoin the negative integers to the natural numbers and zero, and attach to them their familiar properties. We shall now call the set of natural numbers positive integers; and the set consisting of positive and negative integers and zero we shall call a set of integers, and denote it by R. Also $a + (-b)$ will be abbreviated as $a - b$.

Our original definition of congruence can now be modified so that we can say that

$$a \equiv b \pmod{m}$$

if m is in R and > 1, if, and only if, $a - b$ is of the form km with k in R, or $a - b$ is a multiple of m. Note also that the result in Problem 1 holds, if R replaces $N[0]$.

The absolute value $|a|$ of an integer a is defined as the non-negative among the integers a, $-a$, and is is evident that

$$|ab| = |a|\ |b|$$

$$|a + b| \leq |a| + |b|.$$

Problem 2. Show that if f & $g \neq 0$ be integers, then there exists unique integers q, r, such that $f = qg + r$ with $0 \leq r < |g|$.

In (1), in our original set of C's, we may put, as we did in our first paper,

$$C_{m+1} = C_1$$

$$C_1 \neq C_k,$$

where k denotes any natural number in the set 2, 3, $\cdots$, m.

In view of Problem 2, we have that any integer $a \equiv r \pmod{m}$ with $0 \leq r < m$. Hence, the set of integers can de divided into m classes having the form $r + mk_r$; $r = 0, 1, \cdots, m - 1$. Hence, there is an isomorphism between what are called residue classes and the set of C's we just described. This may otherwise be expressed by saying that

$$C_a \longleftrightarrow a$$

$$C_a + C_b = C_{a+b} = C_s \quad (0 \leq s < m), \quad a + b \equiv s \pmod{m};$$

$$C_a C_b = C_{ab} = C_t \quad (0 \leq t < m), \quad ab \equiv t \pmod{m}.$$

The C's are called *residue classes* modulo m.

So far in this paper we have extended the original definition of indeterminate from N to $N[0]$. We may then extend it to R and give without proof

Theorem 4. *If* a_i, x, *belong to* R *and* $a_0 + a_1 x + \cdots + a_n x^n = 0$, *then* $a_i = 0$; $i = 0, 1, \cdots, n$.

When we previously defined the degrees of Type I and Type II polynomials in N we stated that the degree of Type III, which was defined as a fixed element of N would not, at that stage, be defined. We shall here define the degree of k, if k denotes an element of R, as zero when k is regarded as a polynomial.

We may then state without proof

Theorem 5. *If all the polynomials indicated have coefficients* $\in R$, *then it is possible to select for a given* $f(x)$, *a* $q(x)$ *and* $r(x)$ *such that*

$$f(x) = m(x)q(x) + r(x),$$

where the degree of $r(x)$ *is less than that of* $m(x)$, *assuming that* $f(x)$ *is of degree* ≥ 1 *and the coefficient of the power of* x *of highest degree in* $m(x)$ *is unity.*

Now if we define an i such that $i^2 = -1$ and assume that it obeys the formal laws of the arithmetic of the rational integers, then

$$(a_1 i + a_0) + (b_1 i + b_0) = (a_1 + b_1)i + a_0 + b_0,$$

$$(a_1 i + a_0)(b_1 i + b_0) = (a_1 b_0 + a_0 b_1)i + a_0 b_0 - a_1 b_1.$$

We also have

$$a_1 x + a_0 + b_1 x + b_0 \equiv (a_1 + b_1)x + a_0 + b_0 \pmod{x^2 + 1},$$

$$(a_1 x + a_0)(b_1 x + b_0) \equiv (a_1 b_0 + a_0 b_1)x + a_0 b_0 - a_1 b_1 \pmod{x^2 + 1}.$$

We then have a correspondence

$$a_1 x + a_0 \longleftrightarrow a_1 i + a_0$$

for any a_0 and a_1 in R. Now suppose

$$a_1 x + a_0 \equiv c_1 x + c_0 \pmod{x^2 + 1},$$

$$a_1 x + a_0 = c_1 x + c_0 + (x^2 + 1)h(x).$$

Assume $h(x)$ is a polynomial in R. Now $h(x) = 0$ since otherwise the polynomial on the right is of degree at least two by Theorem IV. Hence, we obtain $a_0 = c_0$, $a_1 = c_1$. Consequently, there is an isomorphism between the congruences modulo $x^2 + 1$ with coefficients in R and the complex numbers of the form $a + bi$, where a and b are in R. Such numbers we shall call *Gaussian integers*. (Cf. Also our account of Cauchy's work in the introduction to this paper where R is replaced by the set of real numbers).

Problem 3. Show that if $a + bi = 0$, then $a = 0$ and $b = 0$; a, $b \in R$.

Problem 4. Show that any polynomial in R is congruent modulo x^2 to an expression of the form $a + bx$; a, $b \in R$. This system modulo x^2 is isomorphic with a set of elements $a + bj$, with $j^2 = 0$, $j \neq 0$; that is, j is a *zero divisor* in the latter system. In general, we call $d(\neq 0)$ a zero divisor if $dk = 0$ with $k \neq 0$.

Problem 5. If we replace x^2 in the last problem by $f(x)$, a polynomial in R, what is necessarily true concerning $f(x)$ in order to obtain a system without zero divisors?

Problem 6. In the algebraic systems we are describing above, an element u is said to be a unit providing there exists another element u_1 in the system such that $uu_1 = 1$. Show that in the system of Gaussian integers the only units are ± 1, $\pm i$.

Problem 7. *Set up* a system of elements to form $a + bi\sqrt{5}$ $(a, b \in R)$ by using the modulus $x^2 + 5$. Show that in this system the only units are ± 1. In the relation $2 \cdot 3 = (1 + i\sqrt{5})(1 - i\sqrt{5})$ each one of the four factors appearing in this equation has the property that it is not divisible by any number of the form $a + bi\sqrt{5}$ except itself or a unit. (a is said to be divisible by b if $a = bc$.)

We now introduce the notion of polynomials with a number of indeterminates $x_1, x_2, \cdots x_s$ and write

$$\text{(10a)} \qquad \sum_{i=0}^{k} c_i x_1^{a_{i1}} x_2^{a_{i2}} \cdots x_s^{a_{is}} = f(x_1, \cdots, x_s)$$

where the c's in R, which will be called a polynomial in these x's with coefficients in R. Any equality involving such polynomials will be assumed to hold for x_j any element in R. The following theorem is given without proof:

Theorem 6. *If* $f(r_1, \cdots, r_s) = 0$ *in* (10a), *then each* $c = 0$.

Introduction of the Rational Numbers by Adjunction: Denote by R the set of integers. We begin by extending the notion of congruence and introducing several new "indeterminates"

$$x_a, x_b, \cdots,$$

such that x_a corresponds to the integer a. We say that

$$\text{(11)} \qquad \begin{gathered} f(x_a, x_b, \cdots, x_k) \equiv g(x_a, x_b, \cdots, x_k) \\ (\text{modd}((ax_a - 1), (bx_b - 1), \cdots, (kx_k - 1)). \end{gathered}$$

if $a, b, \cdots, k$ are arbitrary non-zero integers and if polynomials ψ_a, $\psi_b, \cdots, \psi_k$ exist in the x's we are using, such that

$$\text{(12)} \qquad f = g + \psi_a(ax - 1) + \cdots + \psi_k(kx - 1).$$

This will be written

$$f \equiv g \pmod{L}.$$

We shall now show that any polynomial f of the type just mentioned is congruent to an expression of the form αx_a for some integer α and for some integer a. We may affect this by using the following relations (Kronecker):

$$\text{(13)} \qquad \begin{aligned} \alpha x_a + \beta x_b &= (\alpha b + \beta a)x_{ab} + \alpha b x_{ab}(ax_a - 1) + \beta a x_{ab}(bx_b - 1) \\ &- (\alpha x_a + \beta x_b)(abx_{ab} - 1) \equiv (\alpha b + \beta a)x_{ab} \pmod{L}, \end{aligned}$$

if ab is in the set $a, b, \cdots, k$.

$$(14)\quad \alpha x_a \cdot \beta x_b = \alpha\beta x_{ab} + \alpha\beta b x_b x_{ab}(a x_a - 1) + \alpha\beta x_{ab}(b x_b - 1) - \alpha\beta x_a x_b(ab x_{ab} - 1)$$
$$\equiv \alpha\beta x_{ab} \pmod{L},$$

if ab is in the set $a, b, \cdots, k$, and

$$\beta = \beta x_1 - \beta(x_1 - 1) \equiv \beta x_1 \pmod{L},$$

if 1 is in the set $a, b, \cdots, k$.
Employing the formula (14), we reduce any power of any x to the required form, consequently any power product; and following this, we may add the power of products by use of (13). We then have also that every non-zero polynomial has an inverse modulo L since

$$a x_b \cdot b x_a = 1 + (b x_b - 1) + b x_b(a x_a - 1)$$
$$\equiv 1 \pmod{L}.$$

We will now show that

$$\alpha x_a \equiv \gamma x_c \pmod{L},$$

if, and only if,

$$(15)\qquad c\alpha \equiv a\gamma \pmod{L},$$

as is seen by multiplying each side of (15) by ac and using $a x_a \equiv 1$, and $c x_c \equiv 1 \pmod{L}$, provided c is in the set $a, b, \cdots, k$; and we may obtain from (15) the preceding congruence by reversing the procedure.

If we start with the set of natural numbers only instead of the set of integers, we may introduce the moduli y and $z+1$ in addition to L and introduce the rational numbers directly from the natural numbers without explicit introduction of zero and the negative integers.

If we now compare this system modulo L with the rational numbers, we note the correspondence between (13), (14), and (15) and the following relations:

$$(13a)\qquad \frac{\alpha}{a} + \frac{\beta}{b} = \frac{\alpha b + \beta a}{ab}$$

$$(14a)\qquad \frac{\alpha}{a} \cdot \frac{\beta}{b} = \frac{\alpha\beta}{ab}$$

The relation

$$\frac{\alpha}{a} = \frac{\gamma}{c}$$

holds if, and only if,

$$(15a)\qquad c\alpha = a\gamma.$$

In these correspondences we have obviously

$$\alpha_a \longleftrightarrow \frac{\alpha}{a} \quad \text{and} \quad = \longleftrightarrow \overset{L}{\equiv},$$

where the symbol on the right stands for congruence modulo L.

Call F the set of rational numbers as just defined and R the set of rational integers and if $m = a/b$, $a \in R$, $b \in R$, then m is said to be negative if ab is negative. This is symbolized by $m < 0$. Otherwise, m is said to be positive, symbolized by $m > 0$.

If $n \in F$, $s \in F$, and $t \in F$, $n > 0$, with $s + n = t$, then we write $t > s$ or $s < t$.

Problem **8**. Prove that the set F is denumerable.

Problem **9**. (Archimedean Property) If $a \in F$, then we may find a $c \in R$ such that $c > a$.

Problem **10**. Extend the result of Problem 1 to F.

Problem **11**. If $a \in F$ and $b \in F$ with $b > a$, then we may determine $k \in F$ such that $a < k < b$.

Problem **12**. Show that if $bd > 0$

$$\frac{a}{b} > \frac{c}{d}$$

if, and only if, $ad > bc$.

ADDENDUM TO OUR FIRST PAPER, (I).

In this section we will make remarks supplementary to the contents of our (I) and quote various criticisms which have been received from mathematicians concerning the contents of (I) together with our reactions to them.

In (I) we discussed in the introduction the fact that very often in high school, algebra is so taught to the students that even a youngster of real native ability for algebra would likely be repelled. This does not seem to happen so often in connection with the teaching of plane geometry. Consequently, it would hardly be advisable to introduce to university freshmen much, if any, of the foundations of algebra. However, the writer has discussed these topics with sophomores with apparently some success, and I think one reason for this is that I did not do anything except try to set up some rules to justify the operations they were already used to in algebra which is one feature of the developments in (I). I have, a number of times in advanced classes, suggested to the students that they forget everything they know about mathematics, since we would try to start from scratch. I doubt if this would be a very good suggestion to make to a sophomore, however.

If we consider introducing to sophomores the idea of adjunction of zero etc., as given in the present paper, we might find difficulties, such ideas being rather sophisticated. In past years I often preceded such discussions by introducing the number pairs, as is usually done in texts on algebra, together with some extra assumptions about substitution involving these pairs, and then take up the ideas of congruence and homomorphism later in order to justify such adjunctions. However, if a student

really grasps these latter ideas, he usually finds it quite easy to pass to the foundations of the classical theory of algebraic numbers, as we have indicated in the present paper.

A reader of (I) may wonder if we have included the Peano axioms in our set of axioms. They seem to be all found in our discussions on pages 236 and 237 of (I) and in Postulate 4 on page 243, with the exception of the Peano axiom usually expressed as follows: "If $a^+ = b^+$, then $a = b$ if a^+ is the consequent of a and b^+ is the consequent of b." The omission of that latter postulate is what enables us to introduce algebras in which the cancellation law of addition does not hold, ((I) p. 247).

One question raised by a reader of (I) was "How do you know, using your set of axioms, that it will not turn out that all the elements

$$C_1, C_2, C_3, \cdots$$

that we start with (I, p. 242 (3)) are equal?" My answer is, "I don't know."

On p. 235 of (I) we make the following statement: "We do not, however, attempt to describe all the possible methods we shall employ in selecting symbols to denote other types of symbols or sets of symbols, so that our description of procedures with symbols is for that reason incomplete, if not for other reasons." This omission on my part has been regretted by several readers; for example, in this connection, Dr. Alonzo Church wrote me as follows: "As to the confusion between being and denoting, I do hope that you will be able to clear this up before the paper is published. Of course such confusion is nothing new in mathematical papers, even the best of them. But precisely because you are undertaking to exercise more than usual care in such matters as the handling of parentheses (among other things) it becomes more important than usual to eliminate the confusion between being and denoting." I intend to keep this matter in mind and hope to be able to clear it up in a later paper.

I have also received the following criticism from several:-

"... I am still unhappy about your definition of combination. ..."

My reply to this is that I am not too happy about either of my definitions given on p. 242 and in footnote 9 on p. 249. I think it quite possible that someone will be able to set up something I should like to recognize as a combination but which would not be covered by my first definition. As to the second, it seems to me that in it we are not taking full advantage of the special character of the things we are talking about and that the whole thing might be much simplified. Also, I am using in the second definition a number of properties of symbols in finite linearly ordered sets which I do not use elsewhere in the article and which I had hoped to avoid the use of.

Possibly a third and more satisfactory definition might be fashioned based on a little different idea. It is fairly obvious that the difficulties in connection with defining a combination nearly all hinge around

the use of parentheses. Perhaps we can start with a combination containing no parentheses and then indicate a procedure by which we could insert parentheses in the finite linearly ordered set in such a way to obtain in each case a set which we wish to regard as a combination, and none others.

NESTED SERIES, COMPUTATION OF SQUARE ROOTS
AND
SOLUTION OF THIRD DEGREE EQUATIONS

Diran Sarafyan

INTRODUCTION

1. If $y = cf(x)$ where c is a real constant number, we will define an nested series with infinitely many terms by:

$$cf\{x + cf[x + cf(x)]\}\ldots$$

Thus, if $y = -2\log x$, then the corresponding nested series is

$$-2\log[x-2\log(x-2\log x)]\ldots$$

Sometimes such a nested series converges to a limit on some interval. In the forthcoming discussions we will see that the meeting point of two objects or points in motion is represented by the limit of a nested series. Thereafter we will give the geomecrical interpretation and finally we will show that the trinomial equations $x^n + Ax + B = 0$ and $x^n + Ax^{n-1} + B = 0$, have real roots given by the limit of nested series. We will discuss particularily 3rd degree equations.

2. Let us consider two points in motion T_1 and T_2 (see fig.1) departing at the same time from two respective points A_1 and A_2, at distance a from each other. If the motion of these two points is uniform, that is if they are moving following the law:

$$s_1 = t \quad \text{and} \quad s_2 = ct \tag{1}$$

or,

$$s_2 = cs_1$$

s_1 and s_2 being distances covered by T_1, T_2, respectively, it is easily demonstrated by algebra that these two points in motion will meet each other if $|c| < 1$ and that x, the distance covered by T_2 until the time of meeting is given by the relation:

$$x = \frac{ac}{1 - c}$$

which is the solution of a first degree equation.

On the other hand, if v_1, the velocity of T_1 is greater than v_2, the velocity of T_2, we see from figure 1, that when T_1 covers the distance a and arrives at A_2, T_2 in its turn covers the distance ca and arrives at A_3. In short following the given law of motion we will have:

$$A_2A_3 = ac$$

$$A_2A_4 = (ac + a)c$$

$$A_2A_5 = [(ac + a)c + a]c$$

$$\cdots \quad \cdots$$

$$x = A_2B = \text{Lim}\,\{[(ac + a)c + a]c + a\}c \cdots \tag{3}$$

These expressions constitute the simplest of all nested series and the last expression 3, constitutes a nested infinite series which in this particular motion we see is equal to the limit of the sum of the geometrical progression:

$$\text{Lim}(ac + ac^2 + ac^3 + ac^4 + \ldots). \tag{4}$$

3. If we have the following law of motion:

$$s_1 = t \quad \text{and} \quad s_2 = t^2$$

the distances covered by T_1 and T_2 would be at the time of meeting, respectively $a + x$ and x. But $s_2 = s_1{}^2$, hence:

$$(a + x)^2 = x$$

or,

$$x^2 + x(2a - 1) + a^2 = 0$$

a quadratic equation whose roots are given by

$$x = \frac{1 - 2a \pm \sqrt{1 - 4a}}{2} \tag{5}$$

Therefore, we see that our points reach each other provided $1 - 4a > 0$. Furthermore we have now in this case (fig. 1) for the movements of T_2 provided that a short time before meeting $v_1 > v_2$ (otherwise the meeting is impossible):

$$A_2A_3 = a^2$$

$$A_2A_4 = (a^2 + a)^2$$

$$A_2A_5 = [(a^2 + a)^2 + a]^2$$

$$\cdots \quad \cdots$$

$$x = A\ B = \text{Lim}\,\{[(a^2 + a)^2 + a]^2 + a\}^2 \ldots \tag{6}$$

This is another nested infinite series and we know it is convergent if $1 - 4a \geq 0$. And as it will be shown later on, between the two roots offered by the formula 5, the smaller positive root must be chosen as equal to the limit of the nested series 6.

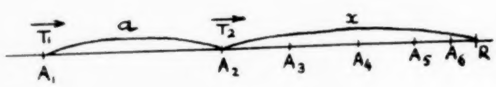


Fig. 1

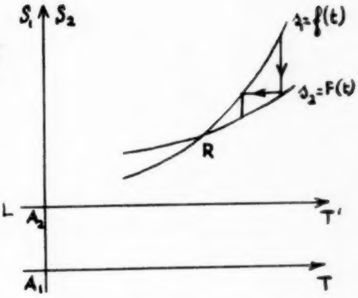


Fig. 3

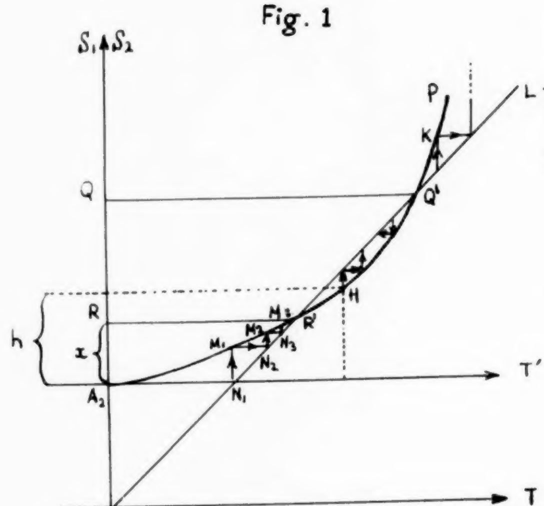


Fig. 2

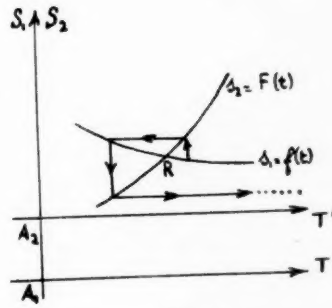


Fig. 4

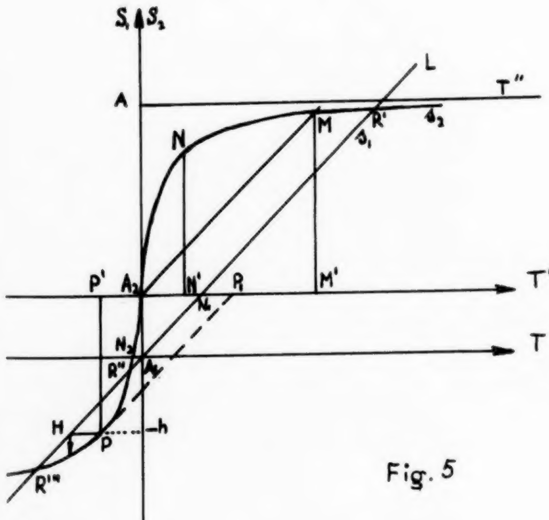
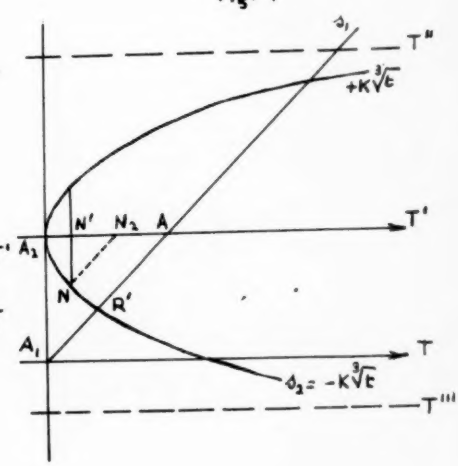


Fig. 5



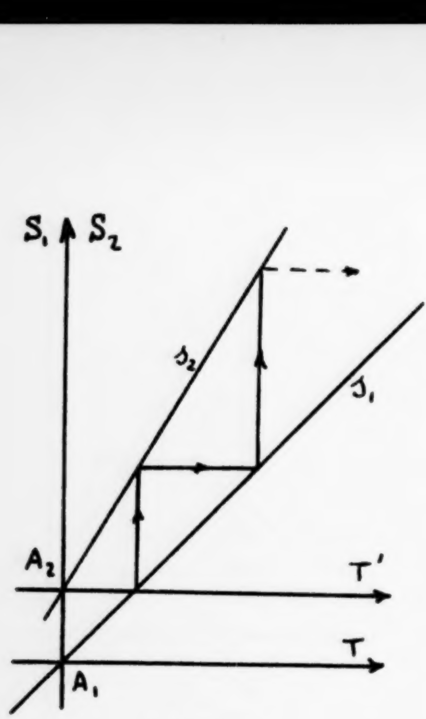


Fig. 7

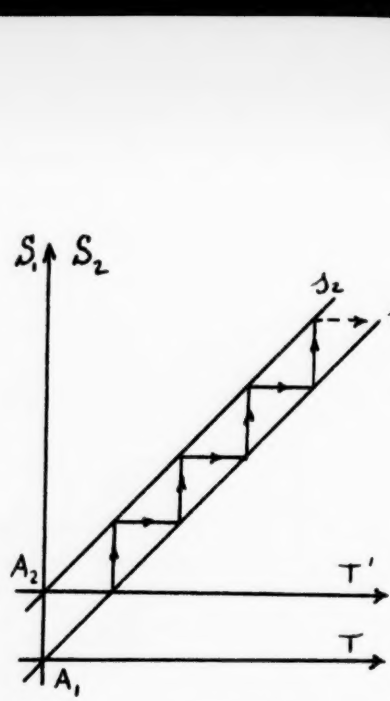


Fig. 8

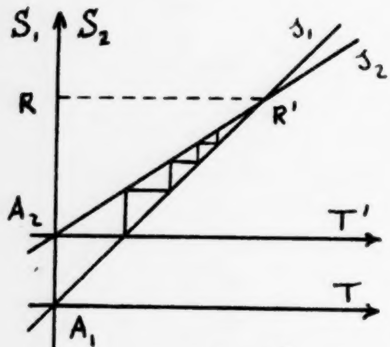


Fig. 9

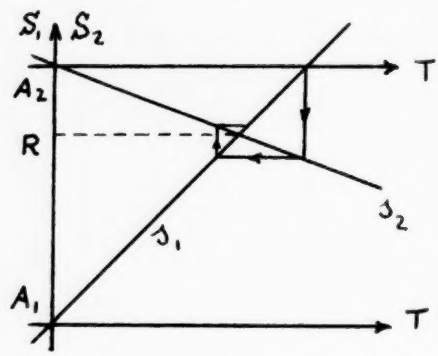


Fig. 10

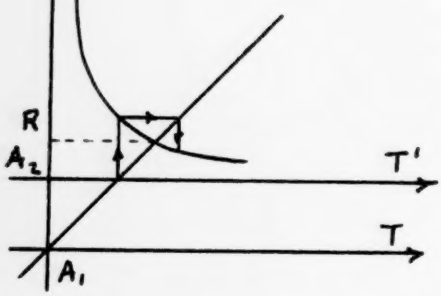


Fig. 11

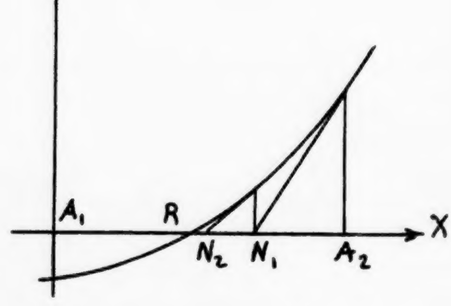


Fig. 12

Therefore we can write:

$$\frac{1 - 2a - \sqrt{1 - 4a}}{2} = \text{Lim } \{[(a^2 + a)^2 + a]^2 + a\}^2 \cdots$$

then,

$$\sqrt{1 - 4a} = 1 - 2a - \text{Lim } \{[(a^2 + a)^2 + a]^2 + a\}^2 \cdots \qquad (7)$$

This formula 7 enables one to compute square roots by successive multiplications and additions. But this is only of theoretical value for it is a slow process.

4. Let us consider the law of motion:

$$s_1 = t \quad \text{and} \quad s_2 = \frac{k}{t}$$

where k is a positive real number.

The distances covered by T_1 and T_2 at the time of meeting are always (see fig. 1) respectively $a + x$ and x. And since we have now $(s_1)(s_2) = k$, the equation giving the distance x covered by T_2 till the time of meeting will be:

$$(a + x)x = k \quad \text{or,} \quad x^2 + ax - k = 0 \qquad (8)$$

whose real roots are given by:

$$x = \frac{-a \pm \sqrt{a^2 + 4k}}{2}$$

provided that $a^2 + 4k \geq 0$ which is always possible if we choose $k \geq -\frac{a^2}{4}$.

By proceeding just as in the previous case we can write:

$$\text{Lim } \cfrac{k}{a + \cfrac{k}{a + \cfrac{k}{a}}} \cdots = x = \frac{-a + \sqrt{a^2 + 4k}}{2} \qquad (9)$$

the right side being the (smaller) positive root of the quadratic equation 11.

From the equality 8. we obtain:

$$\sqrt{a^2 + 4k} = a + 2 \text{ Lim } \cfrac{k}{a + \cfrac{k}{a}} \cdots \qquad (10)$$

which makes the extraction of roots possible through only division and addition.

For instance, if we make $a^2 + 4k = 5$, the arbitrary choice of $k = \frac{1}{4}$ gives us $a = 2$. Hence the equality 10 gives

$$\sqrt{5} = 2 + 2 \text{ Lim } \cfrac{\frac{1}{4}}{2 + \cfrac{\frac{1}{4}}{2}} \cdots$$

With the first division we obtain as approximate value of

$$\sqrt{5} = 2 + \tfrac{1}{4} = 9/4 = 2.25$$

We find next,

$$\sqrt{5} = 2 + 4/17 = 38/17 = 2.2353$$

Thus the first operation gives us a two-digit accuracy and the second a three-digit accuracy (corresponding to a relative error of about 3/10000) as compared to $\sqrt{5} = 2.236$.

This method of computing the square roots by the formula 10 appears to be one of the best of all known methods. While the formula 9 enables one to make the rapid transformation of any real number into an infinite variety of continued simple fractions, or if the continued fraction is given, it makes it possible to see whether or not the series is convergent, and if so, the quick computation of its limit.

II. SOLUTION OF ALGEBRAIC EQUATIONS

5. We saw that the limit of a nested infinite series, corresponding to various laws of motion, represents a root of an equation of either first or second degree. But for the first and second degree equations, our algebraic rules and formulas are extremely simple and moreover they give us exact values. We have no interest in substituting for them an approximate value of a limit. However for algebraic equations of higher than second degree, our algebraic rules and complicated formulas either fail (Cardan's formula, case of three distinct real roots) or we have no direct formula at all and we must apply other rules based on fumbling, testing or on trials (Horner's and Newton's methods etc.).

Consequently for these equations of a degree higher than the second, every time that it is possible, the use of the limit of a nested series as a root of such an equation, becomes if not the best, a good way of computation.

Now, it is advisable that we study first the geometrical or graphical interpretation of nested series.

6. *Geometric Interpretation of Nested Series* — In order to make the present discussion clearer, we will consider once more the law of motion:

$$s_1 = t \quad \text{and} \quad s_2 = t^2$$

The distance covered by T_2 till the time of meeting we know is given by the limit of the nested series

$$\text{Lim } \{[(a^2 + a)^2 + a]^2 + a\}^2 \cdots \tag{6}$$

As figure 2 shows, we will relate thereafter the motion of T_1 to the system of axes S_1A_1T and the motion of T_2 to the system S_2A_2T' always with

$A_1A_2 = a$. Evidently $s_1 = t$ is represented by the straight line A_1L whose slope is equal to one (inclination 45°) and $s_2 = t^2$ is represented by the parabola A_2P. It is obvious that the projection on the axis A_2S_2 of the points of the parabola will indicate the covered distances by T_2, while the projection on the axis A_1S_1 of the points of the straight line, will indicate the covered distances by T_1. Therefore the intersection of P and L, which is the point R' when projected on A_2S_2, will indicate the point of meeting R of T_2 and T_1, supposed to move following the line A_1A_2.

It can be now easily understood that when we write the following nested sequence:

$$a^2;\ (a^2+a)^2;\ [(a^2+a)^2+a]^2;\ \cdots\ ;\ \text{Lim}\ \{[(a^2+a)^2+a]^2+a\}^2\ \cdots$$

we are on the parabola successively on the following points

$$M_1;\quad M_2\quad;\quad M_3\quad;\ \cdots\ ;\quad R'$$

As it is shown on the graph, we have located these points by arrows, departing perpendicularly to the axes of "t", and progressing from the point N_1 of the straight line, step by step to the points $M_1N_2M_2N_3M_3\ \cdots\ R'$.

As far as Q' the other point of intersection, representing the greater root of the quadratic equation is concerned, as the arrows on the graph show it is not a limit unless we reverse our proceeding and adopt the opposite direction; for with the previously adopted direction either we fall back toward R' or we are going farther and farther toward another point which in this particular case is situated at infinity. And it was just for this reason that we stated previously the smaller positive root must be chosen as corresponding to the limit of the nested series.

In other words T_1 can not any more reach T_2 the latter's velocity from now on, becoming greater and greater than that of T_1. It becomes evident that the reverse would happen, that is, T_2 would reach T_1. Our preceding nested series corresponded to $s_2 = t^2 = (s_1)^2$, now, the inverse function being $s_1 = \sqrt{s_2}$ it is natural that the new nested series will correspond to this latter relation. And in fact, by drawing the arrows (fig. 2) and departing from any point beyond R' such as H of the curve P, it can be easily established that the other meeting point represented by A_2Q is equal to:

$$A_2Q = \text{Lim}\ \sqrt{\sqrt{\sqrt{h-a}-a}-a}\ \cdots \tag{11}$$

Therefore when around a meeting point, instead of the usual $v_2 < v_1$, reverse $v_2 > v.$ is happening, then as the relations 6 and 11 show, the nested series and the mathematical operations also are inversed. Besides, if we are working with arrows on a graph, the direction of these arrows also must be inversed. In short, if the usual inequality of velocities is inversed, then everything must be inversed too.

Now considering that in a case of $s = f(t)$ as discussed here, we have:

$$\text{velocity} = \frac{ds}{dt} = \text{slope of } f(t)$$

that is, velocity and slope are represented by the same function, then we can announce the following theorem with respect to graphs:

7. *Theorem of Intersection Concerning Stable Limits*—Every time around a meeting point, if the slope of the curve s_2 is inferior in absolute value to that of s_1 then departing from a convenient point of s_1 and progressing step by step, we will attain a limit which is the point of intersection representing the limit of the nested series (fig. 3). On the contrary, that is, if the slope of s_2 is greater than that of s_1, there is no limit (fig. 4) unless we reverse or inverse our process and start our progression from the curve of greater slope in absolute value.

Note that since we know the relationship between $v_1 = \frac{ds_1}{dt} = 1$ and $v_2 = \frac{ds_2}{dt}$, this theorem is not indispensable, we can always work on a small diagram, like that represented by fig. 1. But in most cases this will be a complicated way; while the application of this theorem which consists in progressing by arrows on a graph, will make the discussion of the problem simple and easier.

8. *Third Degree Equations*—Let us study the following law of motion:

$$s_1 = t \quad \text{and} \quad s_2 = k\sqrt[3]{t} \tag{12}$$

where k is a constant positive number.

First we have to show that there is a meeting point, that is, (see figure 5 where $A_1A_2 = a$), the straight line s_1 intersects the curve s_2. Furthermore, we must see if around the meeting point $v_2 < v_1$ and if so, then proceeding just like in paragraphs 2 and 3 working with diagram or figure 1, we can state that the distance covered by T_2 till the time of the meeting is;

$$\text{Lim} \quad k\sqrt[3]{a + k\sqrt[3]{a + k\sqrt[3]{a}}} \cdots$$

Now, the derivative of s_2 is

$$\frac{ds_2}{dt} = \frac{k}{3\sqrt[3]{t^2}}$$

We see from this expression that, when t = $+\infty$, $\frac{ds_2}{dt} = 0$ which means that the asymptote or the tangent AT'' to the curve s_2 at infinity, is a parallel line to A_1T or A_2T'. But from a point there is only one parallel to a straight line, therefore the line s_1 issued from A_1 cannot be parallel to AT''. Consequently s_1 intercepts AT''. But the curve s_2 is approaching AT'', therefore the line s_1 must also intercept it.

From A_2 let us draw now A_2M a parallel line to s_1. Then by construction the angle $T'A_2M$ being equal to 45°, we have $A_2M' = M'M$. The corresponding value of t for this point M' is given by:

$$t = k\sqrt[3]{t} \quad \text{or} \quad t = \sqrt{k^3} = A_2M' \tag{13}$$

But, A_2N' the value of t for the point N where the slope of s_2 is equal to one, is given by:

$$\frac{k}{3\sqrt[3]{t^2}} = 1$$

Then, we obtain:

$$t = \sqrt{\left(\frac{k}{3}\right)^3} = \frac{k}{3}\sqrt{\frac{k}{3}} = A_2N' \tag{14}$$

Comparing the equalities 13 and 14 we see that $A_2N' < A_2M'$. We can therefore state that, on the curve s_2 departing from the origin A_2, we arrive first at N (slope = 1) then at M and finally at R'. And since the slope of s_2 continuously decreases, we conclude that the slope of s_2 at R' is less than one, which means that $v_2 < v_1 = 1$.

Consequently we can now state that the limit of the nested series

$$x = \text{Lim}\ k\sqrt[3]{a + k\sqrt[3]{a}} \cdots$$

is equal to the smallest positive root of the algebraic equation representing the point of meeting. The equation in this case is:

$$x^3 - (k^3)x - (k^3)a = 0 \tag{15}$$

which is derived from the equality (fig. 1):

$$k\sqrt[3]{a + x} = x$$

But the equation 15 is of third degree, therefore it may have two other roots. Indeed, considering the negative branch of s_2 (fig. 5) we see that we have two more roots x_3 and x_2 corresponding to the points of intersection R''' and R'' provided that we choose $a = A_1A_2 = A_2N_1 < A_2P_1$, the point P_1 being obtained by the line drawn through P ($|\text{slope}| = 1$) parallel to the line s_1.

But in absolute value we see that:

$$A_2P_1 = P'P_1 - P'A_2 = N'N - A_2N' = \frac{2}{3}k\sqrt{\frac{k}{3}}.$$

Therefore the equation 15 will have three real roots if

$$a \le \frac{2k}{3}\sqrt{\frac{k}{3}}$$

the case of equality corresponding naturally to double roots equal to

$$P'P = x_2 = x_3 = -k\sqrt{\frac{k}{3}}$$

Concerning these two roots, we immediately see on the graph (fig. 5) that x_3 corresponds to the usual branch with $v_2 < v_1$ while x_2 corresponds to $v_2 > v_1$ which is an inverse case.

Applying now the theorem of stable limits and departing from the point $H(-h = -k\sqrt{\frac{k}{3}})$ or any other point beyond H toward L', we easily find (steps and arrows almost omitted on fig. 5):

$$x_3 = \text{Lim}\; k\sqrt[3]{a + k\sqrt[3]{a + k\sqrt[3]{a + (-h)}}} \cdots$$

where $-h \leq -k\sqrt{\frac{k}{3}}$ or $h \geq k\sqrt{\frac{k}{3}}$.

For x_2, considering the inverse function of $s_2 = k\sqrt[3]{t} = k\sqrt[3]{s_1}$ which is $s_1 = \frac{s_2^3}{k^3}$ and departing from the point N_2 (ordinate $-a$) of s_2, we similarly find

$$x_2 = \text{Lim}\left\{\frac{1}{k^3}\quad\left[\frac{1}{k^3}\,(-a^3) - a\right]^3 - a\right\} \cdots$$

But reduced third degree equations are in the form:

$$x^3 - \alpha x - \beta = 0 \quad \text{with} \quad \alpha > 0,\ \beta > 0 \tag{16}$$

Comparing the equations 15 and 16, we see that

$$k = \sqrt[3]{\alpha} \quad \text{and} \quad a = \frac{\beta}{\alpha}$$

Substituting these values of k and a in all the above established relations, we can state:

The equation 16 has always one positive root given by

$$x_1 = \text{Lim}\sqrt[3]{\beta + \alpha\sqrt[3]{\beta + \alpha\sqrt[3]{\beta}}} \cdots$$

and if $\beta \leq \frac{2}{3}\alpha\sqrt{\frac{\alpha}{3}}$, it has two more roots which are

$$\text{if} \quad \beta = \frac{2}{3}\alpha\sqrt{\frac{\alpha}{3}} \qquad x_3 = x_2 = -\sqrt{\frac{\alpha}{3}}$$

$$\text{if} \quad \beta < \frac{2}{3}\alpha\sqrt{\frac{\alpha}{3}} \qquad x_3 = \text{Lim}\sqrt[3]{\beta + \alpha\sqrt[3]{\beta + \alpha(-h)}} \quad \text{with} \quad h \geq \sqrt{\frac{\alpha}{3}}$$

$$x_2 = \text{Lim}\left[\frac{(\frac{-\beta}{\alpha})^3 - \beta}{\alpha}\right] \cdots$$

Furthermore, in the equation $x^3 - \alpha x - \beta = 0$, substituting x by $-x$, we obtain $x^3 - \alpha x + \beta = 0$. Then once more we state

Each of the equations

$$x^3 - \alpha x \mp \beta = 0$$

has one root represented by

$$x_1 = \pm \text{Lim} \sqrt[3]{\beta + \alpha \sqrt[3]{\beta + \alpha \sqrt[3]{\beta}}} \cdots$$

besides this, each of these equations has two other roots which might be double or distinct roots as follows

$$\text{if} \quad \beta = \frac{2}{3}\alpha\sqrt{\frac{\alpha}{3}} \qquad x_3 = x_2 = \mp\sqrt{\frac{\alpha}{3}} \tag{17}$$

$$\text{and if} \quad \beta < \frac{2}{3}\alpha\sqrt{\frac{\alpha}{3}} \qquad x_3 = \pm \text{Lim} \sqrt[3]{\beta + \alpha\sqrt[3]{\beta + \alpha(-h)}} \quad \text{with} \quad h \geq \sqrt{\frac{\alpha}{3}} \tag{18}$$

$$x_2 = \pm \text{Lim} \left[\frac{\left(\frac{-\beta}{\alpha}\right)^3 - \beta}{\alpha} \right] \cdots \tag{19}$$

where the first operation giving the first approximation starts for x_3 with $-h$ while for x_2 starts with $\frac{-\beta}{\alpha}$.

If the given motion was:

$$s_1 = t \quad \text{and} \quad s_2 = -k\sqrt[3]{t}$$

we would obtain for the representation of s_2 a curve which is symmetric to the previously considered curve, with respect to the A_2T' axis (fig. 6).

The equation representing the point of meeting on the axis A_2S_2 will be now

$$x^3 + (k^3)x + (k^3)a = 0$$

Naturally, its *only* negative root would be given by the limit of the nested series:

$$x = \text{Lim} -k\sqrt[3]{a - k\sqrt[3]{a - k\sqrt[3]{a}}} \cdots$$

for the branch of s_2 where its slope is less than one in absolute value. We must therefore consider the following three cases:

1st Case — the point of intersection R' is on the branch of s_2 of $|\text{slope}| < 1$ which corresponds to $A_2N_2 < A_2A = a$,

2nd Case — the point R' is on the branch of s_2 of $|\text{slope}| = 1$ which corresponds to $A_2N_2 = A_2A = a$,

3rd Case — the point R' is on the branch of s_2 of $|\text{slope}| > 1$ which corresponds to $A_2N_2 > A_2A = a$,

the point N_2 being obtained by a parallel line drawn from the point N of s_2 corresponding to the $|\text{slope}| = 1$.

We have already calculated the value of A_2N' as being equal to

$$A_2N' = \frac{k}{3}\sqrt{\frac{k}{3}}$$

We also know that in absolute value

$$N'N = k\sqrt{\frac{k}{3}}$$

which is equal to $N'N_2$. Therefore the length of

$$A_2N_2 = A_2N' + N'N_2 = \frac{k}{3}\sqrt{\frac{k}{3}} + k\sqrt{\frac{k}{3}} = \frac{4}{3}k\sqrt{\frac{k}{3}}$$

Now proceeding just the same way as we did previously in establishing our formulas 17-19, we will reach to the following conclusion:

Each of the 3rd degree reduced equations:

$$x^3 + \alpha x \pm \beta = 0$$

has only one root, which is

$$\text{if} \quad \beta > \frac{4}{3}\alpha\sqrt{\frac{\alpha}{3}} \qquad x = \pm \operatorname{Lim} \sqrt[3]{-\beta - \alpha\sqrt[3]{-\beta - \alpha\sqrt[3]{-\beta}}} \;\cdots$$

$$\text{if} \quad \beta = \frac{4}{3}\alpha\sqrt{\frac{\alpha}{3}} \qquad x = \mp\sqrt{\frac{\alpha}{3}}$$

$$\text{if} \quad \beta < \frac{4}{3}\alpha\sqrt{\frac{\alpha}{3}} \qquad x = \pm \operatorname{Lim}\left[\frac{\left(\frac{+\beta}{-\alpha}\right)^3 + \beta}{-\alpha}\right] \;\cdots$$

The preceding four equations $x^3 - \alpha x \mp \beta = 0$ and $x^3 + \alpha x \pm \beta = 0$ can be represented by $x^3 + Ax + B = 0$ where A and B are positive or negative real numbers. Then we can condense all our statements and formulas as follows:

The reduced 3rd degree equation

$$x^3 + Ax + B = 0 \tag{20}$$

has in all cases at least one real root; it may have two other real roots when $A < 0$. These roots are:

when $A < 0$

$$x_1 = \operatorname{Lim} \sqrt[3]{-B - A\sqrt[3]{-B}} \;\cdots \tag{21}$$

$$\text{and if} \quad \left(\frac{B}{2}\right)^2 + \left(\frac{A}{3}\right)^3 = 0 \qquad x_3 = x_2 = \frac{B}{|B|}\sqrt{\frac{-A}{3}} \tag{22}$$

$$\text{and if } \left(\frac{B}{2}\right)^2 + \left(\frac{A}{3}\right)^3 < 0 \quad x_3 = \operatorname{Lim} \sqrt[3]{-B-A\sqrt[3]{-B-A\left(\frac{Bh}{|B|}\right)}} \cdot \quad \text{with } h \geq \sqrt{\frac{-A}{3}} \tag{23}$$

$$x_2 = \operatorname{Lim} \left[\frac{\left(\frac{B}{-A}\right)^3 + B}{-A}\right] \cdots \tag{24}$$

when $A > 0$

$$\text{and if } -\left(\frac{B}{4}\right)^2 + \left(\frac{A}{3}\right)^3 < 0 \quad x = \operatorname{Lim} \sqrt[3]{-B - A\sqrt[3]{-B}} \cdots \tag{25}$$

$$\text{and if } -\left(\frac{B}{4}\right)^2 + \left(\frac{A}{3}\right)^3 = 0 \quad x = \frac{-B}{|B|}\sqrt{\frac{A}{3}} \tag{26}$$

$$\text{and if } -\left(\frac{B}{4}\right)^2 + \left(\frac{A}{3}\right)^3 > 0 \quad x = \operatorname{Lim} \left[\frac{\left(\frac{B}{-A}\right)^3 + B}{-A}\right] \cdots \tag{27}$$

where the first operation giving the first approximation starts,

for the formulas 21 and 25 with $\sqrt[3]{-B}$

for the formula 23 with $\frac{(Bh)}{|B|}$

for the formulas 24 and 27 with $\left(\frac{B}{-A}\right)$

Furthermore, as far as the relation $(B/2)^2 + (A/3)^3 < 0$ is concerned, it is derived from $\beta < 2/3\ \alpha \sqrt{\alpha/3}$ of formula 18. In fact this latter relation can be also written

$$\frac{\beta^2}{4} - \frac{\alpha^3}{27} < 0$$

And since we are in a case of $A < 0$, then $\alpha = -A$; on the other hand $\beta^2 = B^2$. Therefore

$$\frac{\beta^2}{4} - \frac{\alpha^3}{27} = \frac{B^2}{4} + \frac{A^3}{27} = \left(\frac{B}{2}\right)^2 + \left(\frac{A}{3}\right)^3 < 0$$

It is worthwhile to mention that this expression is also found in Cardan's formula for the same reduced 3rd degree equations, in the form

$$\sqrt{\frac{B^2}{4} + \frac{A^3}{27}}$$

which makes the solution of the equation impossible unless complex numbers are introduced. Thus when Cardan's or Tartaglia's formula fails in a such an important case as is the case of three real distinct roots, instead the above formulas give their value easily and rapidly.

In order to make their memorization easy we must point out that the formulas 21 (or 25) and 24 (or 27) are inverse expressions with respect to each other. In all these formulas each of radicals or brackets etc., represents an operation.

As far as the required computations are concerned, they may be worked out with logarithms. But in most cases, where a three-digit accuracy is judged perfectly acceptable, the slide rule can be used advantageously. Thus, an experienced person in slide rule computations, can find, through its use and applying the above indicated method, the approximate value of a root of a 3rd degree equation, in less than one minute.

9. *Examples:*

Example 1: Solve the equation $x^3 - x - 6 = 0$.

Solution: We immediately see that $A < 0$ and $\left(\frac{B}{2}\right)^2 + \left(\frac{A}{3}\right)^3 = \left(\frac{6}{2}\right)^2 - \left(\frac{1}{3}\right)^3$ evidently is not negative; therefore we have only one root which is given by:

$$x = \text{Lim}\ \sqrt[3]{-B - A\sqrt[3]{-B}} \cdots = \text{Lim}\ \sqrt[3]{6 + \sqrt[3]{6}} \cdots$$

since $-A = 1$ and $-B = 6$.

Now, by using the logarithmic tables, we find at;

1st operation		$\sqrt{6}$	$= 1.8171$
2nd operation	$\sqrt{6 + 1.8171} =$	$\sqrt{7.8171}$	$= 1.9846$
3rd operation	$\sqrt{6 + 1.9846} =$	$\sqrt{7.9846}$	$= 1.9987$
4th operation	$\sqrt{6 + 1.9987} =$	$\sqrt{7.9987}$	$= 1.9999$

Since the difference is very small between 1.9987 and 1.9999 we therefore end our operations and accept

$$x = 1.9999$$

as the approximate value of our limit which is also the root of the equation.

Note that the exact value of x is 2. Therefore we committed the following relative errors respectively at 1st, 2nd, 3rd and 4th operations:

about 9%; about 0.7%; about 0.065% and 0.005% = 1/20,000.

These results show how fast and efficient this method is.

Example 2: Solve the equation $x^3 + 12x + 13 = 0$.

Solution: $A > 0$ hence there is only one real root.

Now, $-(B/4)^2 + (A/3)^3 = -(13/4)^2 + (12/3)^3$ is evidently not negative, hence the root is given by

$$x = \text{Lim}\left[\frac{B + \left(\frac{B}{-A}\right)^3}{-A}\right] \cdots$$

where the first operation starts with $(B/-A) = (13/-12) = -1.08$.

Continuing we find at

$$\text{2nd operation } \frac{13+(-1.08)^3}{-12} = \frac{-11.73}{12} = -0.98$$

$$\text{3rd operation } \frac{13+(-0.98)^3}{-12} = \frac{-12.06}{12} = -1.00 \text{ almost}$$

$$\text{4th operation } \frac{13+(-1.00)^3}{-12} = \frac{-12.00}{12} = -1.00$$

Then,

$$x = -1.00$$

the computation being made with a slide rule. Note that the exact value of this root is -1 too.

10. *Equations Higher Than 3rd Degree* — All our previous statements and findings can be extended to equations of higher degree, provided that they are of the form:

$$x^n + Ax + B = 0$$

Particularly when n is an odd positive integer, then all previously established formulas in connection with $x^3 + Ax + B = 0$ are valid, provided that the following changes are made in them:

a: Substitute 3 by n (except 3 of the symbol x_3)

b: Substitute 2 by $n - 1$ (except 2 of the symbol x_2)

c: Substitute 4 by $n + 1$.

Thus for instance, the relation

$$\left(\frac{B}{2}\right)^2 + \left(\frac{A}{3}\right)^3 \leq 0$$

becomes

$$\left(\frac{B}{n-1}\right)^{n-1} + \left(\frac{A}{n}\right)^n \leq 0 \tag{28}$$

which will characterize the condition under which an equation of an odd degree but of the type $x^n + Ax + B = 0$, can have three real roots. On the contrary the equation has only one real root.

When the trinomial equation is of an even degree, in ways similar to those of 3rd degree equations but introducing also the case of $a < 0$ (which means interchange of origins A_1 and A_2) we will find that it has:

$$\text{two real roots if } \left(\frac{B}{n-1}\right)^{n-1} - \left(\frac{A}{n}\right)^n \leq 0 \tag{29}$$

no real roots if relation (29) is > 0.

Now, we see that the formulas 28 and 29 can be expressed by a single relation

$$\Delta = -\left(\frac{-A}{n}\right)^n + \left(\frac{B}{n-1}\right)^{n-1} \leq 0$$

on condition that $n > 1$.

Hence it can be stated that the trinomial equation $x^n + Ax + B = 0$ when of an odd degree has three real roots if $\Delta \leq 0$ and one root if $\Delta > 0$; and when of even degree it has only two roots if $\Delta \leq 0$, on the contrary, that is if

$$\Delta = -\left(\frac{-A}{n}\right)^n + \left(\frac{B}{n-1}\right)^{n-1} > 0 \tag{30}$$

we have no real roots at all. We can call Δ the discriminant of trinomial equation*.

Thus, one may think, the equation

$$x^{1,000,000} + x + 1 = 0$$

which is of one millionth degree, may have one million real roots and in reality it has not even one single real root, because $\Delta > 0$.

Note that for $n = 2$ we have

$$\Delta = -\left(\frac{-A}{n}\right)^n + \left(\frac{B}{n-1}\right)^{n-1} = -\frac{A^2}{4} + B$$

and if

$$\Delta = -\frac{A^2}{4} + B > 0$$

or,

$$A^2 - 4B < 0$$

the equation $x^2 + Ax + B = 0$ has no real roots. This condition although very well known, often is not suspected of being a particular case of the general expression 30, which can also be established by the use of algebra alone.

11. By considering other laws of motion we will find other formulas for the solution of equations. Thus, for instance for the 3rd degree reduced equations of the type $x^3 \pm \alpha x^2 \mp \beta = 0$, we find:

$$x = \pm \operatorname{Lim} \sqrt{\frac{\beta}{\alpha + \sqrt{\frac{\beta}{\alpha}}} \cdots}$$

which does not require any extraction at all of cubic roots.

Furthermore, a great many algebraic equations by means of suitable substitutions can be transformed into our case $x^n + Ax + B = 0$. In particular, to the equations of the types: $x^p + Ax^m + B = 0$ when $p/m = n$ is an integer (let $x^m = y$, then $x^p = y^n$) and $x^n + Ax^{n-1} + B = 0$ (let $x = 1/y$) our method can be applied readily.

*These statements can be established also by a similar direct way, by breaking the equation $x^n + Ax + B = 0$ into $Y = x^n$ and $y = -Ax - B$ and then considering the conditions of their intersection, etc..

SECTION III

12. *Geometric Representation of Geometric Series and Simple Continued Fractions* — We have shown that the geometric series and the simple continued fractions were nested series. Since these two series are the best known we will give their geometric representation in accordance with what has been established in paragraphs 2, 4, 6 and 7.

The Geometric Series — We saw that to the law of motion

$$s_1 = t \qquad \text{and} \qquad s_2 = kt$$

when a is the separating distance of A_1 and A_2 the origins of the two systems of axes, corresponds the geometric progression:

$$a + ka + k^2a + \cdots + k^na + \cdots$$

The following four cases are worthy of consideration:

$1 < k$	the geometric series is divergent	(fig. 7)
$1 = k$	the geometric series is divergent	(fig. 8)
$0 < k < 1$	the geometric series is convergent	(fig. 9)
$-1 < k < 0$	the geometric series is convergent	(fig. 10)

Continued Fractions — To the law of motion

$$s_1 = t \qquad \text{and} \qquad s_2 = k/t$$

corresponds the simple continued fraction

$$\cfrac{k}{a + \cfrac{k}{a}} \quad \cdots$$

We give the geometric representation of the case $k > 0$ in figure 11, to which a convergent series always corresponds (Exercise, discuss the case $k<0$).

Furthermore, we established the relation

$$\sqrt{N} = \sqrt{a^2 + 4k} = a + 2 \operatorname{Lim} \cfrac{k}{a + \cfrac{k}{a}} \quad \cdots \tag{10}$$

which enables one to compute the square roots.

But we can also compute the square roots through Newton's method. In fact, let N be the number of which the square root is desired. We can write:

$$x = \sqrt{N} \qquad \text{or} \qquad x^2 - N = 0. \tag{31}$$

Let us set:

$$f(x) = x^2 - N$$

We know that if a is a number in the neighborhood of the root of the equation (31), then we have a better approximation for this root x, by the application of Newton's method (fig. 12)

$$x = a - \left[\frac{f(a)}{f'(a)}\right] = a - \left[\frac{a^2 - N}{2a}\right] = \frac{1}{2}\left[a + \frac{N}{a}\right].$$

Now, concerning the computation of square roots through the formula (10), if a is given as our first approximation, then as 2nd approximation we have:

$$a + 2k/a \tag{32}$$

But since $N = a^2 + 4k$, then

$$k = (N - a^2)/4 \tag{33}$$

Substituting (33) in (32) we have

$$a + \frac{2}{a}\left[\frac{N - a}{4}\right] = \frac{a^2 + N}{2a} = \frac{1}{2}\left[a + \frac{N}{a}\right]$$

In other words, we find the unexpected result that the 2nd approximation of $\sqrt{N}$ obtained through continued fractions and Newton's method are identical.

If by I_n and II_n we designate the nth approximation for continued fractions and for Newton's method respectively, we find in a similar way:

$$a = I_1 = II_1; \quad I_2 = II_2; \quad I_4 = II_3; \quad I_8 = II_4; \quad \cdots$$

This shows that though both methods are related to each other, the second method is a faster process. However, it can be shown that as the degree of algebraic equations increases, Newton's method gradually loses its efficiency while the implicit series become more efficient. In fact, already for the 3rd degree case nested series furnish faster convergence.

The iteration scheme for a general function f given by Mr. Sarafyan on the first page of his article has been considered from different points of view by various authors. In particular we call the reader's attention to the following references: B. H. Bissinger, "*A Generalization of Continued Fractions*", Bulletin Am. Math. Soc. v. 50 (1944) p. 868; O. W. Richard, same bulletin, v. 53 (1947), together with an earlier article by Everett Also in the Bulletin, v. 52 (1946) and cited in Bechard's paper; J. Ginsburg, Scripta Mathematica, v. 11 (1945) p. 340. The author also points out that his fig. 9 appears on p. 27 of the syllabus for "Certificat d'Eatudes Sup. de Math. Generales", Paris, by G. Bouligand to provide a picture of a convergent geometric progression. -- Editor.

TEACHING OF MATHEMATICS

Edited by

Joseph Seidlin and C. N. Shuster

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, testd and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, *as a teacher*, are interested, or questions which you would like others to discuss, should be sent to Joseph Seidlin, Alfred University, Alfred, New York.

THE MOST IMPORTANT OBJECTIVE IN THE TEACHING OF MATHEMATICS

Frank Gisonti

As the end of each school year approaches I ask myself a question. If my answer is "no" then I know that I have missed the mark and failed as a teacher. It is a question all of us should ask ourselves.

The question concerns the objectives in teaching mathematics in the elementary school. Before reading on----please stop. List your objectives for teaching mathematics. I have asked this of many teachers and their lists have included the ability to perform computations; the ability to estimate and approximate; an understanding of the structure of our decimal system, etc. etc. These and other objectives listed are admirable goals.

However, there is one objective which I find preponderantly excluded. One that is so important, yet so neglected; so evident but so overlooked; so integral a part of the goals that without it, I say, the other goals may become hindrances and stumbling blocks.

What am I talking about? What objective do I have in mind? It is simply stated in one question: Have the children in my class enjoyed learning mathematics? If they haven't, I have failed. They may perform computations with facility, estimate and approximate with equal skill---but how have they been taught and what has evolved? Have they "learned" in addition, to shudder at the sight if an algorism, to detest the word "mathematics" and its significance? Has the subject matter been presented in such a manner that they abhor school and their teacher? Have they acquired the tenderness that leads to cold clammy hands and an increased pulse rate when confronted by a problem in mathematics? Have they developed attitudes which are expressed by avoiding the subject and by looks of apprehension and signs of restlessness when it is approached? If these are the attained "by-products" outcomes of the subject "matter" objectives then I say, I have failed!

Yes, let's teach mathematicsbut in so doing let us redirect our aims. Surely, we want our children to know their subject but let us keep in mind that children will remember what they enjoy learning and, what is most important, they will want to learn more of it. Let us aim for objectives which make the child, not the subject matter, the center of the target. His needs, his feelings, his emotions are of paramount importance. At the end of this school year, ask yourself, "Have the children in my class enjoyed learning mathematics?"

I hope your answer is "yes".

About 40 years ago, David Eugene Smith, one of the rare great teachers of teachers of mathematics, wrote a textbook on the Teaching of Geometry (still the most exciting literary achievement in that field — though for many years out of print). In it he marshalls in a masterly fashion all the genuine and spurious objectives in the study of demonstrative geometry, and then he balances all of them against one: "Demonstrative geometry has survived in our school curricula because it's fun". About that time Edward L. Thorndike was experimenting with, developing, and enunciating Laws of Learning. One of them, The Law of Effect, was destined to dominate for many years, for better or for worse, the psychology of learning. Satisfyingness (enjoyment) accelerates learning. Among our philosophers and practitioners in Education, history records many who subscribed to "whistle while you work", or the enjoyment of learning. On the other hand, perhaps at no time in the history of schooling was there a lack of supporters of Mr. Dooley's much quoted dictum: "It don't matter what you learn them (pupils), so it's hard and they don't like it". Not all of the Dooleyites were or are vicious or sadistic. Many of them believe, "with all their might" that *enjoyment of learning may lead to the learning of enjoyment.*

In the last 30-odd years we have witnessed a compounding of confusion in the interpretation of "good and evil" in work and play. I recall a seriously argued proposal that our workbooks be renamed playbooks, since learning was rapidly becoming more like play and less like work; since, in fact, psychologically(?) it would be more desirable to associate (identify) learning with play rather than with work. However, throughout the history of schooling there were some effective teachers who claimed that learning is hard labor but that the right kind of hard labor could be fun or play or enjoyment. I should therefore like to change the emphasis in Mr. Gisonti's question from: "Have the children in my class *enjoyed* learning mathematics"? to: "Have the children in my class enjoyed *learning* mathematics?"

May we have — we earnestly solicit — an "avalanche of comments" from our readers. J.S. Ed.

MISCELLANEOUS NOTES

Edited by

Charles K. Robbins

Articles intended for this department should be sent to Charles K. Robbins, Department of Mathematics, Purdue University, Lafayette, Ind

GREEK?? MATHEMATICS

Many mathematical historians have wondered how Greek mathematics rose from its primary stages at about 600 B.C. to the polished system represented by Euclid (300 B.C.) in three short centuries. Recent (1945-9) investigations of certain Babylonian clay tables from Plimpton Library at Columbia University, Yale University's Babylonian Collection, The British Museum and The Louvre have shed light on this remarkably rapid rise in mathematical knowledge. These collections contained numerous undeciphered cuneiform tablets.

Many of these tablets containing columns of numbers had been classified as " commercial accounts ", and were thought for years to be of little interest, so that no attempt at translation had been made. Imagine the amazement of a recent translator who discovered that these tablets actually contain mathematical tables and text books! Some of the tablets from about 2000 B.C. contain mathematics of a much more advanced stage than we had previously dreamed that the Babylonians knew. The cuneiform tablets give problems involving area and volume, surveying, engineering construction, simple and compound interest, and division-of-estate problems. Apparently the Babylonians even knew how to solve quadratic equations! The theorem we usually attribute to Pythagoras was known to the Babylonians more than 1000 years before Pythagoras' time. Perhaps most amazing of all is that the tablets reveal a remarkably efficient numerical trigonometry.

Oklahoma University — Mathematics Letter

INTERNATIONAL CONGRESS OF MATHEMATICIANS 1954

The International Congress of Mathematicians 1954 will be held in Amsterdam from September 2nd to September 9th under the auspices of " Het Wiskundig Genootschop " (The Mathematical Society of the Netherlands). It is the sincere hope of the " Wiskundig Genootschap " that the Congress 1954, which will be open to all mathematicians from all parts of the world, will be a fertile international gathering.

There will be two categories of membership in the Congress: regular members (members) who will be entitled to participate in the scientific and social activities of the Congress and to receive the Proceedings of the Congress, and associate members who, accompanying regular members of the Congress, do not participate in the scientific programme nor receive the Proceedings, but will be entitled to many of the privleges of membership.

The fees to be paid by members and by associate members have not yet been definitely fixed but presumably they will not exceed the amount of fifty guilders (about $14.-) for members and twenty guilders ($5.50) for associate members.

Those who wish to attend the Congress are requested to communicate their name (with degrees, qualifications etc.) and full address to the secretariat as soon as possible. They will receive a more detailed communication which will be sent out in the course of 1953.

Amsterdam, February 1953
2d Boerhaavestraat 49 The Organizing Committee.

ORIGIN OF RADIAN

Information about when and by whom the radian concept was originated would be appreciated, as would also references dealing with the origin of this concept. Mathematical histories seem to be deficient in this connection.

Ed Bergdal

Ellensburg, Washington

Can someone furnish this information? If so, please communicate with the editor of this department.

SLIDE RULE CRITERION

Can a slide rule be made for calculations with the formula $z = f(x,y)$? It is necessary and sufficient that there be a function of z, $S(z)$ for which x and y can be separated, so that we have $S(z) = X(x) + Y(y)$. If we take first and second x and y partial cross-derivatives, we get

$$S''(z)\ \frac{\partial z}{\partial y}\frac{\partial z}{\partial x} + S'(z)\ \frac{\partial^2 z}{\partial x\ \partial y} = 0$$

whence

$$S''(z)/S'(z) = -\ \frac{\partial^2 z}{\partial x\ \partial y} \div \left[\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\right]$$

If we multiply both sides by dz, on the left we have the differential of $\ln[S'(z)]$, and in order to find $S(z)$ it is essential that

$$\left[\frac{\partial^2 f}{\partial x\ \partial y} \div \frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\right]\ \left[\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy\right]$$

be an exact differential. This gives us as the necessary criterion

$$\frac{\partial}{\partial y}\left[\frac{\partial^2 f}{\partial x\ \partial y} \div \frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial x}\left[\frac{\partial^2 f}{\partial x\ \partial y} \div \frac{\partial f}{\partial x}\right]$$

This criterion is illustrated by the following examples.

The formula $z = xy(x+y)$ does not satisfy this criterion, because it leads to $-2/(x+2y)^2$ and $-2/(2x+y)^2$ which are not equal. A nomographic chart can be made for it, but not a slide rule.

The two integrations by which S is found, when it exists, introduce constants of integration. What is their effect in generalizing the ordinary slide rule which computes $z = xy$? We get $S''/S' = -1/z$, whence $\ln S' = -\ln(z/A)$, and $S = A\ \ln Bz$. This can be written

$$z = A\ \ln B_1\, x + A\ \ln B_2\, y$$

where $B_1 B_2 = B$. This shows that nothing new is added to our resources in designing slide rules for multiplication. The A merely determines the length of the dekad -6, 8, 10, 20 inch scales are common — and the B's determine where the scale begins -- usually at 1, but sometimes at π as in the F or *fractus* scale.

As an example of choosing a particular constant of integration, take $z = \sqrt{x^2+y^2}$. This gives $S''/S' = -1/z$. For the first integral we may take either $\ln S' = \ln z$ or $\ln S' = \ln z + \ln 2$. The latter gives the simpler function $S = z^2 = x^2 + y^2$.

The arithmetic, geometric, and harmonic mean formulas all yield simple slide rules. For $z = (x+y)/2$, S is the same as z. For $z = \sqrt{xy}$, $S''/S' = -1/z$, and $S = (1/2)\ \ln x + (1/2)\ \ln y$. For $z = 2xy/(x+y)$, $S''/S' = -2/z$ and $S = -1/z = -1/2x\ -1/2y$.

From $z = xy + x + y$, we get $S''/S' = -1/(z+1)$ and $S = \ln(z+1) = \ln(x+1) + \ln(y+1)$. From $z = \sin(ax+by)$, $S''/S' = z/(1-z^2)$ and $S = \arcsin z = ax + by$.

Tufts College William R. Ransom

CURRENT PAPERS AND BOOKS

Edited by

H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to H. V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.

A Change in Book Reviews

The increasing number of books coming to us for reviewing has made it impossible to secure and publish the reviews, in the usual way, within a reasonable length of time after the publication of the books. Hence we have devised a new procedure which we believe will remove some of the present hurdles and still give the characteristics of the books, which are of most interest to our readers.

From now on we will publish two types of announcements if furnished by the publishers and submitted to us within a year before or after the publication of the book. These announcements must be signed by their authors, require no hand-setting and be subject to the same editorial criticism as other material.

1) Announcements of texts on standard courses, limited to 120 words.

2) Longer releases on books in new fields or very unusual treatments of established courses, limited to 500 words.

Two such announcements follow:

The Teaching Of Secondary Mathematics. By Claude H. Brown, New York, Harper and Brothers, 1953, 388 pp. $4.00.

The conflict of opinion regarding the nature of mathematics and its place in education has led to so much confusion and controversy that the average teacher of secondary mathematics is likely to be somewhat bewildered about what to teach and how to teach it. This book is based on the assumption that teaching method is individual; that a teacher who has an adequate knowledge of the subject matter of his field; who is familiar with the needs, aspirations, and potentialities of his students; who understands the nature of the learning process; and whose philosophy of education is appropriate to a democratic society will be able to devise methods and techniques suited to his particular situation.

Central Missouri State College — Claude H. Brown

A set of techniques developed to meet practical industrial and economic problems is described in *An Introduction to Linear Programming,* published

in February, 1953. The authors are A. Charnes, W. W. Cooper, and A. Henderson, all at the Carnegie Institute of Technology.

The book introduces first the theory of linear programming from the economic point of view. Then, the latter half covers mathematical theory.

Working details of planning and analysis are covered at two levels — that of the individual firm as well as industry-wide and economy-wide activity. The book is further devoted to the best means of planning this activity where it involves such simultaneous factors as market conditions, profit possibilities, capacity limits, time and production requirements, quality considerations, and balance between sales and production.

John Wiley & Sons, Inc. Richard Cook

I am enclosing a notation made some months ago in regard to the solution of the system of equations given on p. 87, vol. 26, no. 2, Nov.-Dec., 1952, Mathematics Magazine, under the general discussion of rounding off errors in the Tausky Todd article. I agree with the authors that the "behavior of this system is blamed on its lack of *condition*" but the ancient and honorable elimination method still serves us well. (This example was set up by T. S. Wilson).

$$10x + 7y + 8z + 7w = 32 \qquad (1)$$

$$7x + 5y + 6z + 5w = 23 \qquad (2)$$

$$8x + 6y + 10z + 9w = 33 \qquad (3)$$

$$7x + 5y + 9z + 10w = 31 \qquad (4)$$

Eliminate x,y in Eq. (2) and (4) $\quad .3z + .5w = .8 \quad$ (5)

Eliminate x in Eq. (1) and (3) $\quad .05y + .45z + .425w = .925 \quad$ (6)

Eliminate x in Eq. (1) and (4) $\quad (1/70)y+(34/70)z + (51/70)w = 43/35 \quad$ (7)

Eliminate x in Eq. (1) and (2) $\quad (1/70)y+(4/70)z+(1/70)w = 3/35 \quad$ (8)

Eliminate z in Eq. (7) and (8) $\quad -3y + 17w = 14 \quad$ (9)

Eliminate z in Eq. (5) and (6) $\quad -.05y + .325w = .275$ or $-2y + 13w = 11 \quad$ (10)

Eliminate y in Eq. (9) and (10) $\quad w = 1$

Rand Corporation
1700 Main St., Santa Monica, California Mrs. Bernice Brown

I am in a good position to respond to Mr. Foster's article, DON'T CALL IT SCIENCE, (Math. Mag., vol. 26, no. 2, March-April, 1953, p. 209) because I also combine professional work in public relations with a creative interest in mathematics.

I think I know what Mr. Foster means. His intention is to correct a tendency to confuse the abstract science of mathematics, which is its own

universe to be explored, with the natural sciences which, particularly in contemporary America, tend to conceive of mathematics as a tool of computation. He is also objecting to the teaching of mathematics as a useful implement or "necessary evil" for a career in the natural sciences, instead of identifying mathematics as a world of its own to be valued for its own sake.

Unfortunately, Mr. Foster hangs his argument upon a total misconception, amounting to an inversion, of the meaning and history of the word "science". By historical tradition and continued usage, mathematics is not only "The Queen of the Sciences", a title used by Eric T. Bell; this subject, which Alfred N. Whitehead once likened to Ophelia in Hamlet ("charming ··· and a little mad"), is also the Queen Mother of the Sciences for all the exact sciences, and the name of science itself, are derived from mathematics.

This point was brought out by the late Charles S. Slichter, one of America's most beloved teachers of pure and applied mathematics, in a delightful essay, *Pilymaths: Technicians, Specialists, and Genius,* where he explains:

"In ancient times, of course, all engineers were scientists, and all scientists were masters of many fields. The etymology of the word *mathematics* illustrates this. *Mathematics* does not mean mathematics; it means science or, more accurately, general science or all science."

Thus, the original name for *science* was *mathematics,* and to declare that mathematics is not a science is, etymologically, an inversion of facts. It would be more appropriate if the natural sciences were deprived of the use of the name, science, which came into being through mathematics.

This is purely a verbal point. A more important point is that, from ancient to modern times, exact science has evolved out of mathematics. As Slichter points out in his essay, many modern engineers earn their living from the theory of moments defined by Archimedes in, be it noted, a purely mathematical paper. Archimedes, like other ancient founders of exact science, conceived that the order of nature was to be understood through pure mathematical ideas, and he performed physical experiments only to fix or to verify these ideas.

Newton worked in the same way to found the theory of universal mechanics. He did not use mathematics as a "serving maid" but as the door to universal knowledge. As he said in the preface to his Principia, he proposed "to subject the phenomena of nature to the laws of mathematics." As far as Newton was concerned, *science* was primarily and dominantly mathematics, and only secondarily experimental and observational.

Einstein's procedure in framing his theory of relativity and unified field theory does not differ in principle from these classical examples. His purpose was and is, like that of Newton, to frame a mathematical theory of natural phenomena in which mathematical order is not servant but master.

There is a vast difference, as the Greek philosophers understood most clearly, between the use of numbers and other mathematical entities for practical computations in commerce, engineering, etc., and the science of

mathematics. Mr. Foster is not clear on this point. Mathematics is a universal science, the original and foundational exact science, and the door to all exact science, Calculation or reckoning is now, as it has been from ancient times, a practical language of quantity which is properly subordinate to the sciences, industries, etc., that it serves.

The science of mathematics is not a system of computation, nor is it a philosophy except as any science or art is its own philosophy. The science of mathematics deals with a universe of order which the various subsciences of mathematics, known and to be discovered, explore according to the laws of that universe.

The distinction between the mathematical universe and the natural or sensory universe appears when we consider the fundamental difference between mathematical reasoning and physical reasoning.

When we isolate the mental field of the human organism from the special conditions of the sensory world, and think with ideal generality as if nature did not exist, the resulting mental operations are mathematical reasoning and the resulting order is pure mathematics. In this method of reasoning we accept no limitation upon generality of thought. The logical order of mathematics is as abstract and general as the ideally isolated field of human intelligence.

When we correlate the ideal order of mathematics with the sensory world, by the use of sense data abstracted from that world, the operations of thought are those of physical reasoning, i.e., mathematical reasoning modified by physical (sensory) conditions, and the resulting order is mathematical physics. In this method of reasoning we organize or pattern special conditions derived from the sensory world within the ideal framework of order derived from mathematics.

The very identity of exact science, in all the basic sciences of nature, issues from and is defined by mathematics. Thus, to say that mathematics is not a science, in the sense that physics or astronomy are sciences, is to say that science is not science, a manifest absurdity.

As a member of the American Mathematical Society, and a reader of the Society's Proceedings and Transactions, may I say that your modest and unpretentious publication consistently maintains a higher level of interest for readers interested in the world of mathematics, rather than in special advances, than these more formal and formidable publications.. Excess of professionalism, in any field, tends to smother a subject beneath formalities of exposition and technical refinements. Surely the living subject of mathematics, in its broadest aspects, needs a representative in America, and it is a sad commentary on the state of our allegedly scientific civilization that this published representative must be printed with utmost economy for a limited circulation.

American Institute of Man
200 East Superior Street
Chicago 11, Illinois

Alexander W. Ebin

Introduction to Metamathematics. By Stephen Cole Kleene, D. Van Nostrand Company, Inc., New York, N. Y., 1952, $x+550$ pp., $8.75.

Under the influence of Hilbert's method for putting the consistency of mathematics on a firm foundation, as a result of the work of the logicistic school (Peano, Whitehead and Russell) and of the intuitionists (Brouwer, Weyl), the formalistic attack upon the problem developed into the discipline which is now known as metamathematics. The most complete systematic presentation of this theory was found, until now, in the books by Hilbert-Ackermann and by Hilbert-Bernays. Professor Kleene's book makes a very important contribution to the literature in this field. Indeed, as far as the reviewer is aware, it is the first book on this subject in English, thus opening the way for a more extensive development than has heretofore been possible. This fact alone would be sufficient to welcome the appearance of this volume and to congratulate both its author and the editorial board of the "University series in higher mathematics" under whose auspices it has been published.

Metamathematics is "a program which makes a mathematical theory itself the object of exact mathematical study" (p. 59). The word "metamathematics" has been formed in analogy with the word "metaphysics", a branch of philosophy which is not concerned with physical phenomena, but which deals with the first principles upon which rests the development of the various sciences. Obviously, such a program requires that the mathematical theory be "formalized", i.e. that "the propositions of the theory should be arranged deductively, some of them, from which the others are logically deducible, being specified as the axioms", and that "all the properties of the undefined and technical terms of the theory which matter for the deduction of the theorems have been expressed by axioms" (p. 59). The principles and methods which underlie such a program are developed in the first eight chapters.

The book consists of four parts. In part I (pp. 3-65) the general problems are presented to which a study of the foundations of mathematics leads. Part II (pp. 69-213) treats various aspects of mathematical logic, formal deduction in Chapter V, the propositional calculus in Chapter VI and the predicate calculus in Chapter VII. In part III (pp. 217-386) the general theory of recursive functions is developed; and part IV (pp. 389-516) takes up additional important topics in mathematical logic, carrying the treatment forward from the point reached on p. 180. The formal text is followed by an extensive bibliography (pp. 517-537), a list of symbols and notations, and a very useful index.

It is clear that we have here a very substantial book, dealing with a theory which has been developed during the last 25 years, which occupies an important place in current research and which has proved to have applications in a variety of directions. This latter fact is likely to cause surprise at first. But when one becomes aware of the fundamental character of the theory, and reflects that, the deeper the roots of a subject, the more broadly the visible growth is likely to spread, the wide applicability of the metamathematical theory will appear as quite natural.

It is impossible to discuss the rich contents of this book in detail; a few remarks must suffice. It may be considered as standing between the formalistic and the intuitionistic schools, in so far as it restricts itself to "methods, called *finitary* by the formalists, which employ only intuitively conceivable objects and performable processes (p. 63). In many instances, definitions and theorems are formulated in two ways, one valid in the formalistic theory and the other in the intuitionistic. A very useful feature of the book is the extensive use of examples. Repeatedly the argument in a proof is interrupted to introduce examples which illustrate the abstract ideas that have been presented up to that point; these illustrations are then carried along as the argument develops. Informal, heuristic discussions are frequently introduced as preliminaries to the presentation of rigorously formal concepts and as preparations for proofs.

Special attention should be called to the full discussion of Gödel's fundamental paper (1931; vol. 38 of the Monatshefte) in Chapters IX and X, to the description in Chapter XIII of the Turing machine and of "Turing's thesis that every function which would naturally be regarded as computable is computable under his definition, i.e. by one of his machines" (p. 376), to Church's λ-definability, to the "word problem for semi-groups" (p. 383) and to Gentzen's affirmative answer to the question as to the existence of a normal form for proofs and deductions in Chapter XV.

Arnold Dresden

Complex Analysis. By L. V. Ahlfors, New York, McGraw-Hill, 1953, 11 + 247 pp., $5.00.

This is an excellent book that reflects the author's broad experience as both a contributor and a teacher of complex variable theory. The principle asset of the book is the spirit of integrity set forth by the author in the preface and rigidly adhered to throughout the text.

The book is divided into six chapters. The first chapter presents the algebra and geometry of complex numbers, the linear fractional transformation and exceptional consideration to the concepts of arguments and roots of complex numbers. The second chapter starts with some elementary functions then a restrictive definition of an analytic function, point set topology necessary to function theory and a stimulating discussion on arcs and closed curves. The chapter ends sith a study of conformal mapping and a descriptive introduction to Riemann surfaces.

Chapter three develops the theory of complex integration. This is based on the theory of real valued and continuous functions over real intervals. Cauchy's theorem and integral formula are first proved for elementary regions which enables the author to derive important results such as Morrera's theorem, taylor's theorem, local mappings, the maximum modulus principle and Schwarz's lemma. Parenthetically, it might have been interesting if the author mentioned the connection between Schwarz's lemma and the first paragraph of this chapter. A proof of the general form of Cauchy's theorem is

presented and is based on the notion of winding number which, in the reviewer's opinion, is neat. The last part of the chapter contains more than an ample amount on the calculus of residues.

The next chapter considers sequences of analytic functions. Uniform convergence, Taylor and Laurent series, partial fractions and infinite products are studied. Applications are then made to entire functions. The last section of the chapter contains an excellent discussion on normal families and the general Riemann mapping theorem. Especially impressive is the discussion on normal families. The fifth chapter is concerned with the Dirichlet problem. It includes topics such as harmonic functions, Poisson's integral formula, Harnach's principle and the powerful Jensen's theorem. The chapter ends with subharmonic functions and the solvability of the Dirichlet problem and with mappings of multiply connected regions.

The last chapter deals with the general structure of an analytic function. Riemann surfaces, analytic continuation and the monodromy theorem are presented. The chapter finishes with a section on algebraic functions and a section on differential equations in the complex domain. Throughout this chapter, as in previous chapters, the author has inserted meaningful exercises that greatly increase the value of the book as a classroom text.

The reviewer's overall estimate of the book can be best expressed by saying that he hopes it will become the accepted text in each class where complex variable theory is studied and where a book is used to outline the course.

Pasquale Porcelli

Elements of the Topology of Plane Sets of Points (2nd ed., reset). By M. H. A. Newman, London and New York, Cambridge University Press, 1951, 214 pages, $4.75.

This is an extremely fine book. It is written so that it can be profitably read by a student knowing nothing about topology provided he has a fairly good background in Mathematics. The book covers a very large body of material. It is suprising to find so much covered in a few pages, yet every step is clearly stated. There are a few places where a few additional words would perhaps help, but everything essential is present.

Chapters 1, 2, 3, form a good introduction to the basic properties of abstract sets, sets in metric spaces, compactness, homeomorphisms, topological properties etc. Chapter 4 is concerned with connected sets, components, locally connected sets, and their properties. Many results are proved, leading up to the wellknown theorems characterizing the arc, simple closed curve, and continuous curve in terms of their topological properties. For example: A necessary and sufficient condition that a space be the continuous image of [0,1] is that it be compact, connected, and locally connected.

In chapter 5 by very neat devices the famous Jordan theorem is proved, that each simple closed curve C separates the plane into exactly two complementary open connected sets, each having the frontier C. Most of the results

of this chapter are extended to Euclidean spaces of any finite dimension — the only departure from the plane in the whole book.

Chapter 6 appealed perhaps most of all to this reader. It contains many results concerning simply connected and multiply connected domains, their complements, their frontiers, mappings of them onto certain standard simple types of sets, uniform local connectivity, and accessibility. In chapter 7, homotopy is considered, with application to orientation of plane curves and the Cauchy integral theorem.

Errors and misprints were almost non-existent, the most obvious one being perhaps in the definition of uniform local connectivity. Exercises are well-chosen. Altogether it is a very rewarding book to read, and provides a good introduction to topology.

University of Saskatchewan Garth H. M. Thomas

Methods of Applied Mathematics. By F. B. Hildebrand, Prentice-Hall, New York, 1952, pp. xi + 523, \$7.75.

Physicists and engineers inevitably find that more and better mathematics leads to unity and economy of thought. In recent years a number of books of the type under consideration have appeared. The selection and treatment of the mathematical material is of necessity determined by the field of applications and the previous training of the intended readers.

The book by Bronwell is intended for students who have had courses in mathematics through the calculus. The book contains discussions of infinite series, elementary functions of a complex variable, differential equations including solution in series and accounts of the Bessel and Legendre functions, Fourier-trigonometric, -Bessel, and -Legendre expansions, the Fourier integral, Laplace transforms, vector analysis, and analytic functions of a complex variable including the evaluation of integrals and conformal mapping. The mathematics merges into the applied field with the treatment of elastic and electric oscillations, Lagrange's equations, the wave equation, heat flow, fluid dynamics, electromagnetic theory, and dynamic stability. There are problems at the end of each chapter, and the answers to some of them are given. The book would be more useful if these problems were more numerous and more comprehensive. The reviewer is of the opinion that the applied part of this book will be valuable to the students for whom it is written. Generally speaking, the mathematical arguments tend to plausibility rather than rigor. The reviewer distrusts the eventual value of the amount of this kind of reasoning which the author uses.

The book by Hildebrand is intended for the somewhat more mature student who would doubtless have covered many of the topics in the book of Bronwell. Here the emphasis is on the mathematics, and the arguments are more complete and satisfying. There are 386 problems of varying difficulty with answers. The four chapters are almost completely self contained and independent. The first chapter treats linear algebraic equations, quadratic and Hermitian forms, and operations with vectors and matrices. The second chapter treats

the calculus of variations including Hamilton's principle and Lagrange's equations. The third chapter combines the presentation of available methods for solving the simpler types of difference equations with a description of the application of finite-difference methods to the approximate solution of problems governed by partial differential equations. The last chapter deals with linear integral equations including exact and approximate methods of solution and the several interpretations of the relevant Green's function.

References for further reading are given at the end of the chapters in both books.

Carman E. Miller

Fundamentals of the Calculus. By Donald E. Richmond, New York, McGraw-Hill Book Company, Inc., 1950, ix + 233 pages, \$3.00.

This is a carefully written brief introduction to calculus designed for liberal arts students who elect one year of mathematics and for students of physics who wish to begin the study of calculus in their freshman year. The seven chapter headings are Numbers, Functions and Graphs, Derivatives, Integration, Logarithmic and Exponential Functions, Complex Numbers and Trigonometry, The Analytic Geometry of the Conics. In view of the purpose of the book, these topics are treated in a surprisingly thorough manner. The author's style is easy and pleasant, and the tone of the book is precision, clarity and rigor.

Unusual features of the book are the definition of $\ln x$ as the solution of $\frac{dy}{dx} = \frac{1}{x}$ which vanishes at $x = 1$ and the use of complex numbers in developing properties of $\sin x$ and $\cos x$. Topics omitted include Rolle's theorem, Mean Value Theorem, L'Hospital's rule, polar coordinates, parametric equations, the techniques of formal integration, and Taylor's formula with remainder. However, polynomials which approximate $\ln(1 + x)$, e^x, $\sin x$, $\cos x$, and Arc $\tan x$ are obtained, together with an estimate of the error in each case.

While it is clear that a calculus text written for freshmen would not be expected to include all the usual topics, the reviewer believes that the omission of the Mean Value Theorem was a mistake. On p. 108 the proof that if $f'(x) \equiv 0$, then $f(x) =$ constant is incomplete. The Mean Value Theorem is needed.

Another theorem that could have been included at little additional cost in space is the fact that the derivative of a differentiable continuous function takes on every value between two values. With the use of this theorem, it is easy to give a complete explanation of example 3 in pp. 94-95. The definition on p. 117 of definite integral is certainly not the customary one. On p. 85 it is not made clear that if $v = 0$, then $v_1 = 0$ for all x_1 in some neighborhood of x. On p. 77, Theorem 8 neglects to assume that P is an interior point.

Omission of the lower limit on an integral sign on p. 139 is the only typographical error noted. The book is well constructed and clearly printed on paper of good quality.

J. M. Hurt

PROBLEMS AND QUESTIONS

Edited by

Robert E. Horton, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 N. Vermont Ave., Los Angeles, 29, California.*

PROPOSALS

175. *Proposed by Jack Winter, Venice, California.*

Reconstruct the following cryptarithm given that each of the letters represents a distinct digit.

$$\begin{array}{r} D\ \ C\ \ A \\ \sqrt{AA\ BC\ CB} \\ {*}{*}\qquad\quad \\ \hline {*}\ {*}{*}\quad\ \ \\ {*}\ {*}{*}\quad\ \ \\ \hline {*}{*}\ {*}{*} \\ {*}{*}\ {*}{*} \\ \hline \end{array}$$

176. *Proposed by W. R. Ransom, Tufts College.*

Let TA, length $2(OT)$, be a tangent at T to the circle whose center is at O. Draw AO, cutting the circle at C. Let M be the midpoint of AC. With C as center draw a circle through M cutting the given circle at P and P'. Prove that this construction gives PP' the side of a regular pentagon inscribed in the circle whose radius is OT.

177. *Proposed by Murray S. Klamkin, Polytechnic Institute of Brooklyn.*

If $w = z^n + a_1 z^{n-1} + \cdots + a_n + b_1/z + b_2/z^2 + \cdots + b_r/z^r$ maps into $|w| = 1$ for $|z| = 1$, show that $a_n = b_r = 0$, $n = 1, 2, 3, \cdots$ and $r = 1, 2, 3, \cdots$.

178. *Proposed by Pedro A. Piza, San Juan, Puerto Rico.*

Find prime numbers x, y, and z, $2000 > z > y > x > 1$ satisfying the Pythagorean equation $(y + 4z - x)^2 + (3y)^2 = (y + 4z)^2$.

179. *Proposed by John M. Howell, Los Angeles City College.*

Seven men are in a room when a fire breaks out and lights go out. They rush to a closet and each gets a hat when he leaves. What is the probability that exactly four get their own hats?

180. *Proposed by Leon Bankoff, Los Angeles, California.*

In a right triangle ABC whose sides are integers, $AC > BC > AB$. The bisector of angle ABC meets AC at D. E is the projection of A upon BD and F is the midpoint of AC. If $EF = 49$ find AB, BC and AC.

SOLUTIONS

Late Solutions

149, 151. *Leo Moser, University of Alberta, Canada.*

A Constant Function

153. [November 1952] *Proposed by John R. Hatcher, Brown University.*

Prove, without using the exponential function $e^{f(z)}$, that if $f(z) = u + iv$ is entire and $u \neq 0$, then $f(z)$ is a constant.

Solution by the proposer. If $u > 0$, consider the function $F(z)$ defined by $F(z) = [f(z) + k^2]^{-1}$, k a real constant. Clearly $F(z)$ is entire and $|F(z)| = 1 \div \sqrt{(u+k^2)^2 + v^2} \leq 1/(u+k^2) < 1/k^2$. Hence $F(z)$ is constant (Liouville's theorem), and so $f(z)$ must be constant. If $u < 0$, consider the function $G(z)$ defined by $G(z) = f(z) - k^2$. Here $|G(z)| = \sqrt{(u - k^2)^2 + v^2} \geq k^2 - u$. Hence $|1/G(z)| \leq 1/(k^2 - u) < 1/k^2$ and again by Liouville's theorem $G(z) = f(z) - k^2$ is a constant.

Also solved by *M. S. Klamkin, Polytechnic Institute of Brooklyn.*

Primitive Pythagorean Triangles

154. [January 1953] *Proposed by Francis L. Miksa, Aurora, Illinois.*

Find primitive pythagorean triangles whose area A contains all the ten digits 1,2,3,4,5,6,7,8,9,0.

Solution by the proposer. Four solutions have been found so far.

Generators		A	B	C	Area
320	49	31,360	99,999	104,801	1,576,984,320
293	122	71,492	70,965	100,733	2,536,714,880
322	83	53,452	96,795	110,573	2,586,943,170
298	179	106,684	56,763	120,845	3,027,851,946

One triangle containing all the digits 1 to 9 was found.

691	2	2,764	477,477	477,485	659,873,214

It is noted that such triangles seem to be very scarce.

Circles About An Equilateral Triangle

155. [January 1953] *Proposed by Leon Bankoff, Los Angeles, California.*

An equilateral triangle is circumscribed by a chain of six equal and successively tangent circles in such a manner that three of the circles

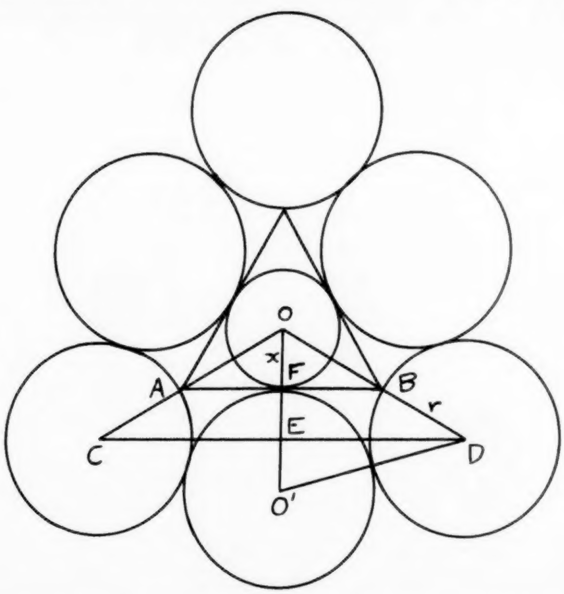
touch the vertices and the other three the midpoints of the sides of the triangle. Show that the inradius of the triangle is equal to the side of a regular decagon inscribed in one of the tangent circles.

I. *Solution by W. B. Carver, Cornell University.* Let C be the center of the inscribed circle of the equilateral triangle, and A and B be the centers respectively of a circle through a vertex and a circle touching an adjacent side. We may take the radius of the inscribed circle of the triangle as the unit of length, and let x be the radius of each of the surrounding circles. Then ABC is a triangle with a $60°$ angle at C and sides of $2x$, $1+x$ and $2+x$. Using the Cosine Law we have $4x^2 = (1+x)^2 + (2+x)^2 - (1+x)(2+x)$ or $x^2 - x - 1 = 0$ giving $x = (\sqrt{5}+1)/2$.

The side of a regular decagon inscribed in a circle of radius r is $r(\sqrt{5}-1)/2$. Hence in the circle of radius $(\sqrt{5}+1)/2$ the side of the inscribed decagon is $(\sqrt{5}+1)(\sqrt{5}-1)/4 = 1$ which proves the theorem.

Solutions essentially similar to this one were submitted by *Sam Kravitz, East Cleveland, Ohio; A. Mac Neish, Chicago, Illinois; O. M. Niersbach, Bell, California; C. W. Trigg, Los Angeles City College and the proposer.*

II. *Solution by Daniel Weiner, Wright Junior College, Chicago, Illinois.*



In the figure let x be the inradius of the equilateral triangle. Then $OB = 2x$ and, by the Pythagorean Theorem, $FB = x\sqrt{3}$. Also $CA/AO = DB/BO = EF/FO$; so $EF/x = r/2x$. Therefore $EF = r/2$ and $O'E = r/2$. In the right triangle EOD; $(2x+r)^2 = (x+r/2)^2 + (ED)^2$ and so $ED = (x+r/2)\sqrt{3}$. In the right triangle $EO'D$ we have

$$(x+r/2)\sqrt{3} = \sqrt{4r^2 - (r/2)^2}$$

$$x + r/2 = r/2\sqrt{5}$$

$$x = r(\sqrt{5}-1)/2$$

But the side, s, of a regular decagon inscribed in a circle of radius r is $r(\sqrt{5}-1)/2$ so $x = s$ and the problem is proved.

Other solutions using synthetic geometry were submitted by *Danny Cooper and Annette Mayhew (Jointly), Alexander Hamilton High School, Los Angeles, California; Elaine Hemenway, Catholic Girls High School, Los Angeles, California and Philip Maclasky, Philadelphia, Pennsylvania.*

A Quadrilateral with Maximum Integral Area

156. [January 1953] *Proposed by E. P. Starke, Rutgers University.*

A quadrilateral has sides of length 2, 3, 4, and 5 in that order. Determine the angle between the sides of length 4 and 5 such that the area shall be the largest possible integer.

Solution by L. A. Ringenberg, Eastern Illinois State College. Let A denote the angle between the sides of length 4 and 5. Let B denote the interior angle opposite to A. Note that A is a positive acute angle. From the Cosine Law we obtain (1) $10\cos A - 3\cos B = 7$. The area of the quadrilateral is given by (2) $K = 10\sin A + 3\sin B$. Then $\dfrac{dK}{DA} = 3\sin B(\cot A + \cot B)$. K is a maximum when $\cot A + \cot B = 0$. That is, when the angles A and B are supplementary (and hence the quadrilateral is cyclic). From equations (1) and (2) we have $K_{max} = 13\sin A$ where $13\cos A = 7$. Thus $K_{max} = \sqrt{120}$. The largest integral K is 10 so from (1) and (2) we obtain $3\sin B = 10 - 10\sin A$. and $3\cos B = 10\cos A - 7$. There are two solutions:

$$A = \text{arc sin}\,(120 \pm 7\sqrt{5})/149$$

which gives the approximate values of:

$$A = 44°27'09''$$

and

$$A = 65°33'47''$$

Several solvers started with the fact that a quadrilateral with fixed sides has a maximum area when it is cyclic. Thus Heron's Formula $S = \sqrt{(s-a)(s-b)(s-c)(s-d)}$ leads to $S = \sqrt{5\cdot4\cdot3\cdot2}$ or $S = \sqrt{120}$ as the maximum area.

Also solved by *Leon Bankoff, Los Angeles, California; W. B. Carver, Cornell University; R. Huck, Baltimore, Maryland; H. I. James, Hampton Institute; Sam Kravitz, East Cleveland, Ohio; C. W. Trigg, Los Angeles City College and the proposer.*

A Factorial Ending In Zeros

158. [January 1953] *Proposed by Leo Moser, University of Alberta, Canada.*

Find all values of r such that no $n!$ (written in decimal notation) can end in exactly r zeros.

Solution by C. W. Trigg, Los Angeles City College. If 2^p and 5^q are the highest powers of 2 and 5 which divide $n!$, clearly $p > q$ and $n!$ ends in q zeros. Furthermore $q = [n/5] + [n/5^2] + [n/5^3] + \cdots$, where $[x]$ denotes the greatest integer in x.

Now as n takes on successive integer values, q also takes on successive integer values, but more slowly, with the exception of those values r which are skipped when $[n/5^i] = (n/5^i)$. Hence we have $r = (6k - 1) + p_1[k/5] + p_2[k/5^2] + p_3[k/5^3] + \cdots$, where $k = 1, 2, 3, \cdots$ and $p_i = 1$ except when $[k/5^i] = (k/5^i)$, when p_i also has the value $(k - 5^i)/k$. Thus the first 31 values of r are: 5, 11, 17, 23, 29, 30, 36, 42, 48, 54, 60, 61, 67, 73, 79, 85, 91, 92, 98, 104, 110, 116, 122, 123, 129, 135, 141, 147, 153, 154, 155.

Also solved by *B. A. Hausman, West Baden College, Indiana* and *C. R. Perisho, Nebraska Wesleyan University.*

The Broken Stick

159. [January 1953] *Proposed by A/2C D. L. Silverman, Patrick A. F. B., Florida.*

A man breaks a stick in two places. What is the probability that he will be able to form a triangle with the three segments?

Solution by W. Funkenbusch, Michigan College of Mining and Technology, Sault Ste. Marie Branch. Let the lengths of the three segments be given by x, y, $L - x - y$, (L a constant). Then clearly:

$$p = \frac{\int_0^{L/2}\int_{L/2-x}^{L/2} dy\,dx}{\int_0^{L}\int_0^{L-x} dy\,dx} = \frac{1}{4}$$

It should be noted that the problem is not new. It is found in *Uspensky's "Introduction to Mathematical Probability"* and also in *Ball's Mathematical Recreations and Essays".*

Bankoff found the problem in the *Hall and Knight, "Higher Algebra", London 1948* and also in *"Solutions of the Problems and Riders proposed in the Senate-House Examinations for 1854", Cambridge: Macmillan and Co. (1854), pages 49-52.*

Also solved by *A. L. Epstein, Cambridge Research Center; Sam Kravitz, East Cleveland, Ohio; B. I. Pfeiffer, Outremont, Quebec; L. A. Ringenberg, Eastern Illinois State College; Milton Scharf, Brooklyn, New York; Dmitri Thoro, University of Florida* and *C. W. Trigg, Los Angeles City College.*

Data For Triskaidekaphobes

160. [January 1953] *Proposed by Victor Thebault, Tennie, Sarthe, France.*

In which of the remaining years of the twentieth century will Friday-the-thirteenth occur most (or least) frequently?

Solution by Leon Bankoff, Los Angeles, California. The table below is readily verified:

WHEN JAN. 1st FALLS ON	Common Year	Leap Year
	Friday-13 will occur in	
Monday	April, July	Sept., December
Tuesday	Sept., Dec.	June
Wednesday	June	March, November
Thursday	Feb., Mar., Nov.	Feb., August
Friday	August	May
Saturday	May	October
Sunday	Jan., Oct.	Jan., Apr., Jul.

In common years Friday-the-thirteenth occurs three times if January 1st falls on Thursday. In Leap Years Friday-the-thirteenth occurs most frequently when January 1st falls on Sunday. From an easily constructed table of New Years Days for the remainder of the century we find that the years 1953, 1956, 1959, 1970, 1981, 1984, 1987 and 1998 will have three Fridays-the-thirteenth.

Friday-the-thirteenth occurs only once in common years which begin on Wednesday, Friday or Saturday and in leap years which begin on Tuesday, Friday or Saturday. So the least number of Fridays-the-thirteenth occur in the years 1954, 1955, 1958, 1960, 1965, 1966, 1969, 1971, 1972, 1975, 1977, 1980, 1982, 1983, 1986, 1988, 1993, 1994, 1997, 1999, 2000.

Also solved by *C. W. Trigg, Los Angeles City College.*

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 95. Find the smallest number with 28 divisors. [*Submitted by M. S. Klamkin.*]

Q 96. Square both 43 and 214 mentally. [*R. R. Abbott in Scripta Mathematica,* **14**, 111, (June 1948).]

Q 97. Solve $ax^2 + bx + c = 0$ without completing the square or using the quadratic formula. [*Submitted by M. S. Klamkin.*]

Q 98. When 12345679 is multiplied by $9 \times k$, $(k = 1, 2, 3, \cdots, 9)$, the product obtained is *kkkkkkkkk*. Show how to find all other numbers $a_1 a_2 a_3 \cdots a_p$ which when multiplied by $b \times k$ yield products of the form $kkk \cdots k$. [*Submitted by Dewey Duncan.*]

Q 99. Evaluate $\int_0^\infty \log x\, dx/(1+x^2)$. [*Submitted by M. S. Klamkin.*]

Q 100. Show that any function is the sum of an odd function and an even function. [*Submitted by Samuel Skolnik.*]

Q 101. Prove that $2^{1/2}+3^{1/3}$ is irrational. [*Submitted by M. S. Klamkin.*]

ANSWERS

A 95. Since $28=2\cdot2\cdot7$, the number must be of the form $2^a3^b5^c$ where $(a+1)(b+1)(c+1)=28$. Thus the number is $2^6\cdot3\cdot5$ or 960.

A 96. Since $a^2=(a+b)(a-b)+b^2$, let $a=43$, $b=7$. Then we have $(43)^3=(50)(36)+7^2=1849$. Now let $a=214$, $b=14$, then $(214)^2=(228)(200)+14^2=45600+196=45796$. This method is useful when a differs but slightly from a multiple of 50 or of 100.

A 97. Let $x=y+h=y-b/2a$. Then we have $ay^2+y(2ah+b)+ah^2+bh+c=0$. Now since $h=-b/2a$, this equation becomes $ay^2-b^2/4a+c=0$, so $y=\pm\sqrt{(b^2-4ac)/4a^2}$ and $x=[-b\pm\sqrt{b^2-4ac}\,]/2a$. [This is the method of Vieta, for example, see D. E. Smith, *History of Mathematics*, Vol. II, (Ginn) 1925, page 449.]

A 98. Just divide the number $111\cdots$ by the odd integers not ending in 5 until the division ends exactly. These quotients times k constitute the desired class. For example: $111\div37=3$, so $37\times3=111$, $37\times6=222$, $\cdots$.

A 99. Let $x=1/y$, then $I=\int_0^\infty\frac{\log x\,dx}{1+x^2}=\int_\infty^0\frac{\log(1/y)(-1/y^2)\,dy}{1+1/y^2}=-\int_0^\infty\frac{\log y\,dy}{1+y^2}=-I$. Thus $I=0$.

A 100. Clearly $f(x)\equiv(1/2)[f(x)+f(-x)]+(1/2)[f(x)-f(-x)]$. Since a function is even if $F(x)=F(-x)$ and odd if $F(x)=-F(-x)$, it follows that the first term of the right hand member is even and that the second term is odd.

A 101. Assume that $2^{1/2}+3^{1/3}=R$, a rational number. Then $3=(R-\sqrt{2})^3=R^2+6R-\sqrt{2}(3R^2+2)$. It follows that $\sqrt{2}$ is rational. Since it is well-known that this is not true, the assumption is false.

FALSIES

A falsie is a problem for which a correct solution is obtained by illegal operations, or an incorrect result is secured by apparently legal processes. For each of the following falsies can you offer an explanation? Send in your favorite falsies.

F 11. Each of the following proper fractions may be simplified by striking out the common digits in the numerators and denominators: 16/64, 19/95, 26/65, 49/98. Why? [*Adapted from R. K. Morley and Pincus Schub, American Mathematical Monthly*, **40**, *425, August 1933.*]

F 12. Locate the error in the following sequence of inequalities: $3>2$, $3\log(1/3)>2\log(1/3)$, $\log(1/3)^3>\log(1/3)^2$, $(1/3)^3>(1/3)^2$, so $1/27>1/9$. [*Submitted by B. K. Gold.*]

F 13. A high-school student solved the linear differential equation $\frac{d}{dx}\frac{dy}{dx} + Py = Q$ for y as if it were an ordinary algebraic one. Under what conditions could this procedure have yielded a correct solution of the differential equation? [*Submitted by M. S. Klamkin.*]

EXPLANATIONS

E 11. These are the only solutions of $(10a + n)/(10n + b) = a/b$. Now $(10a+n)/(10b+n) = a/b$, $(10n+a)/(10n+b) = ab$, and $(10n+a)/(10b+n) = a/b$ have no solutions, so the four proper fractions are the only ones with two-digit numerators and denominators which can be simplified by illegitimate cancellation.

E 12. The logarithm of any number between 0 and 1 is negative, so multiplication by log (1/3) in inequality 2 reversed the inequality.

E 13. The student cancelled the d's and obtained $y = Qx^2/(1+Px^2)$. If this is to be a correct solution of the differential equation, then $d^2y/dx^2 = yx^2$, so $y = ax^{(1+\sqrt{5})/2} + bx^{(1-\sqrt{5})/2}$, where a, b are arbitrary constants. Hence $Q = (1+Px^2)\left[ax^{(\sqrt{5}-3)/2} + bx^{-(\sqrt{5}+3)/2}\right]$. [For a similar problem see, ***THIS MAGAZINE***, 25 113, (Nov. 1951)].